Nonlinear elliptic equations with perturbed symmetry

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Abstract

In this paper, we are interested in studying multiplicity of solutions for nonlinear elliptic equations with perturbed symmetry. Existence of infinitely many solutions of superlinear problems with perturbed symmetry was considered by several mathematicians in 1980's under some restrictive growth conditions on both the unperturbed nonlinear term and the perturbing term which is fixed. While we have a small parameter ε to drive the perturbing term, we only need very weak growth conditions on the unperturbed nonlinear term and the perturbing term. We prove that the equations have as many solutions as prescribed when $|\varepsilon|$ is suitably small. We give a further extension of the classical result of Berestycki and Lions [ARMA, 82 (1983), 347-375] and we also improve the famous symmetric mountain pass theorem due to Ambrosetti and Rabinowitz [JFA, 14 (1973), 349-381]. A key ingredient of our proof is to find an infinite number of *nondegenerate critical values* of the unperturbed energy functionals.

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1 Introduction

It is well known that in many cases even functionals have multiple critical points, and this fact applied to nonlinear differential equations with symmetry yields multiple solutions ([1, 27]). *A natural question to ask is: What happens when such a functional is subjected to a perturbation which destroys the symmetry? Some special cases of this question have been studied and while progress has been made, there are not yet satisfactory general answers* (quoted from Rabinowitz's monograph [27, P. 61]).

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In 1980's, existence of infinitely many solutions of superlinear elliptic equations with perturbed symmetry of the form

$$-\Delta u = |u|^{p-1}u + f(x) \ \text{ in } \Omega, \quad u \in H_0^1(\Omega), \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $N \geq 2$, $f \in L^2(\Omega)$, and $p \in (1, p_N)$ for some $p_N < (N+2)/(N-2)$, was obtained by Bahri [2], Bahri and Berestycki [3], Bahri and P.-L. Lions [4], Rabinowitz [26], and Struwe [29]. It is believed that the result should be true for all $1 < p < (N+2)/(N-2)$ (see [27, P. 69] and [4, P. 1028]), but this is still an open problem. For related results in this direction, see also [11, 19] for superlinear equations and [12, 15, 16, 19] for sublinear equations.

In this paper, we use a unified idea to prove existence of multiple solutions for the following three types of elliptic equations in the whole $\mathbb{R}^N$ with destroyed symmetry. The first is the nonlinear scalar field equation

$$\begin{cases} -\Delta u = f(u) + \varepsilon g(u) & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.2}$$

where $f, g \in C(\mathbb{R}, \mathbb{R})$ and f is odd. The second is the quasilinear Schrödinger equation

$$\begin{cases} -\Delta u + \varepsilon \left(\operatorname{div}\left(G_\xi(x, u, \nabla u)\right) - G_t(x, u, \nabla u)\right) + V(x)u = f(x, u) & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.3}$$

where $V \in C(\mathbb{R}^N, \mathbb{R})$, $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, $f(x,t)$ is odd in t, and $G = G(x, t, \xi) \in C^1(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N, \mathbb{R})$. The third is the nonlinear Choquard equation

$$\begin{cases} -\Delta u + \omega u - \phi_u u + \frac{1}{\varepsilon} g(x, \varepsilon u) = 0 & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases} \tag{1.4}$$

where $\omega > 0$, $\phi_u \in D^{1,2}(\mathbb{R}^3)$ is the unique weak solution of the equation $-\Delta \phi = u^2$ in $\mathbb{R}^3$. In these equations, $\varepsilon \in \mathbb{R}$ is a small parameter with the convention that if $\varepsilon = 0$ then the perturbation term $\frac{1}{\varepsilon} g(x, \varepsilon u)$ in (1.4) is zero.

Here, the pattern of symmetry breaking is not the same as that in (1.1). While we have a small parameter ε to drive the perturbing terms, we allow general superlinear nonlinearity and very general symmetry breaking terms $g(t)$, $G(x, t, \xi)$ and $g(x, t)$ in our results.

For equations (1.2), (1.3), and (1.4), under very general conditions we shall prove that for any given $m \in \mathbb{N}$ there exist at least m solutions provided that $|\varepsilon|$ is sufficiently small (with the understanding that the perturbation term is 0 in (1.4) if $\varepsilon = 0$). A key ingredient in our approach is to find an infinite number of *nondegenerate critical values* of the unperturbed energy functionals. These critical values are also called essential values (see Section 2 for the definition) and have the feature that they continue to exist when the functionals are perturbed. In our results, we do not assume oddness of $g(t)$ or $g(x, t)$ in t or evenness of $G(x, t, \xi)$ in t. Moreover, we do not impose any control on

$g(t)$ or $g(x,t)$ or $G(x,t,\xi)$ for $|t|$ or $|\xi|$ large. That is to say, $g(t)$ and $g(x,t)$ can be of any growth in t and $G(x,t,\xi)$ can be of any growth in $|t|$ and $|\xi|$.

In what follows, we denote $2^* = 2N/(N-2)$ for $N \geq 3$ and $2^* = +\infty$ for $N = 1, 2$. Equation (1.2) is a perturbation of

$$-\Delta u = f(u), \quad u \in H^1(\mathbb{R}^N). \tag{1.5}$$

In the celebrated paper [7], H. Berestycki and P.-L. Lions proved that equation (1.5) has infinitely many solutions when $N \geq 3$ and f satisfies the almost optimal assumptions:

(f_1) $f \in C(\mathbb{R}, \mathbb{R})$ and $f(t)$ is odd;

(f_2) $-\infty < \varliminf_{t\to 0} f(t)/t \leq \varlimsup_{t\to 0} f(t)/t = -\ell < 0$;

(f_3) $\varlimsup_{t\to +\infty} f(t)/t^{2^*-1} \leq 0$;

(f_4) there exists $\zeta > 0$ such that $F(\zeta) = \int_0^\zeta f(t)\,dt > 0$.

Our first main result is the following theorem which asserts that under assumptions (f_1)–(f_4) together with very weak assumptions on g, (1.2) has as many solutions as prescribed if $|\varepsilon|$ is sufficiently small.

Theorem 1.1. *Let $N \geq 3$ and (f_1)–(f_4) be satisfied. Suppose $g \in C(\mathbb{R}, \mathbb{R})$ and*

$$\varlimsup_{t\to 0} |g(t)|/|t| < +\infty. \tag{1.6}$$

Then for any $m \in \mathbb{N}$ there exists $\varepsilon_m > 0$ such that (1.2) has at least m distinct radial solutions provided that $|\varepsilon| \leq \varepsilon_m$.

In Theorem 1.1, we only assume g to be continuous and satisfy (1.6). Neither a condition on g for $|t|$ near ∞ nor oddness of g is needed. Theorem 1.1 extends the main result of H. Berestycki and P.-L. Lions [7] since if $\varepsilon = 0$ then Theorem 1.1 is exactly the main result in [7]. We also point out that, in a suitable functional framework as in [7] our approach to prove Theorem 1.1 provides an infinite number of *nondegenerate critical values* of the unperturbed functional (see Lemma 3.2), and this result seems to be convenient in view of other possible applications.

Using a change of unknown as in [9, 21], we can transform certain types of quasilinear elliptic equations into (1.2) so that results as consequences of Theorem 1.1 can be obtained for quasilinear elliptic equations. The next corollary is just an example in this direction.

Corollary 1.2. *Let $\alpha > 1$. Assume the same conditions as in Theorem 1.1 except that the inequality in (f_3) is replaced by*

$$\varlimsup_{t\to +\infty} f(t)/t^{(2+\alpha)2^*/2-1} \leq 0.$$

Then for any $m \in \mathbb{N}$ there exists $\varepsilon_m > 0$ such that the quasilinear elliptic equation

$$-(1+|u|^\alpha)\Delta u - \frac{1}{2}\alpha|u|^{\alpha-2}|\nabla u|^2 u = f(u) + \varepsilon g(u) \quad \textit{in } \mathbb{R}^N \tag{1.7}$$

has at least m distinct radial solutions provided that $|\varepsilon| \le \varepsilon_m$.

Proof. Set $h(t) = \int_0^t \sqrt{1+|s|^\alpha}\, ds$ and make the transformation $v = h(u)$. Then (1.7) is converted into

$$-\Delta v = \frac{f(h^{-1}(v)) + \varepsilon g(h^{-1}(v))}{\sqrt{1+|h^{-1}(v)|^\alpha}} \quad \text{in } \mathbb{R}^N,$$

which is in the form of equation (1.2). Using properties of h^{-1} (see [9, 21]), it is easy to verify that (f_1)–(f_4) are satisfied by $\frac{f\circ h^{-1}}{\sqrt{1+|h^{-1}|^\alpha}}$ and (1.6) is satisfied by $\frac{g\circ h^{-1}}{\sqrt{1+|h^{-1}|^\alpha}}$. We conclude by Theorem 1.1. □

Equation (1.3) is a perturbation of the semilinear Schrödinger equation

$$-\Delta u + V(x)u = f(x,u), \quad u \in H^1(\mathbb{R}^N). \tag{1.8}$$

To state our result on (1.3), we assume

(V) $V \in C(\mathbb{R}^N, \mathbb{R})$, $V(x) = V(|x|)$, $\alpha_0 := \inf_{x\in\mathbb{R}^N} V(x) > 0$, $\beta_0 := \sup_{x\in\mathbb{R}^N} V(x) < +\infty$;

(F_1) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, $f(x,t) = f(|x|,t)$ is odd in t;

(F_2) $f(x,t) = o(t)$ as $t \to 0$, uniformly for $x \in \mathbb{R}^N$;

(F_3) there exist $C > 0$ and $2 < p < 2^*$ such that, for $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$,

$$|f(x,t)| \le C(1+|t|^{p-1});$$

(F_4) there exists $\mu > 2$ such that, for $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$,

$$F(x,t) := \int_0^t f(x,s)\, ds \le \frac{1}{\mu} f(x,t)t;$$

(F_5) there exist $x_0 \in \mathbb{R}^N$ and $t_0 > 0$ such that $F(x_0,t_0) > 0$;

(G) $G, G_{\xi_i} \in C^1(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, $G(x,0,\xi) \equiv 0$, $G(x,t,\xi) = G(|x|,t,|\xi|)$ and for any $R > 0$ there exists $C_R > 0$ such that if $x \in \mathbb{R}^N$, $|t| \le R$, $|\xi| \le R$, and $1 \le i, j \le N$ then

$$|G_t(x,t,\xi)| \le C_R|t|, \quad |G_\xi(x,t,\xi)| \le C_R(|t|+|\xi|),$$

and

$$\Big|\frac{\partial^2 G(x,t,\xi)}{\partial x_i \partial \xi_j}\Big| + \Big|\frac{\partial^2 G(x,t,\xi)}{\partial t \partial \xi_j}\Big| + \Big|\frac{\partial^2 G(x,t,\xi)}{\partial \xi_i \partial \xi_j}\Big| \le C_R.$$

Note that (V) and (F_1)–(F_5) are classical assumptions in dealing with Schrödinger type equations. Under these assumptions, a result due to Strauss [28] states that (1.8) has infinitely many radial solutions. This result can be proved via several different approaches. Especially, it can be proved using the Ambrosetti-Rabinowitz symmetric mountain pass theorem from [1] (see also [27]) or using the fountain theorem as in [31, Theorem 3.12].

Compared with (1.8), equation (1.3) is much more complex due to the presence of the quasilinear terms. It has only a formal variational structure

$$J_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right) dx - \int_{\mathbb{R}^N} F(x,u)\, dx - \varepsilon \int_{\mathbb{R}^N} G(x,u,\nabla u)\, dx.$$

This functional is invariant with respect to the $\mathbf{O}(N)$ action on $H^1(\mathbb{R}^N)$. However, it is not well defined on any Sobolev space since there is no global control on the growth of G, nor is it an even functional since we do not assume G to be even in t. The control we imposed on the growth of G allows G to be of any growth rate for $|t|$ or $|\xi|$ near ∞.

The second main result is for (1.3) and is stated below.

Theorem 1.3. *Let $N \geq 2$. Assume (V), (F_1)–(F_5) and (G). Then for any $m \in \mathbb{N}$ there exists $\varepsilon_m > 0$ such that (1.3) has at least m distinct radial solutions provided that $|\varepsilon| \leq \varepsilon_m$.*

Theorem 1.3 extends the result of Strauss [28] mentioned above since when $\varepsilon = 0$ Theorem 1.3 reduces to that result, cf. [31, Theorem 3.12]. It is also worthwhile to mention that our proof of Theorem 1.3 is based on a remarkable refinement of the classical symmetric mountain pass theorem due to Ambrosetti and Rabinowitz; see Theorem 4.1 which asserts that an even functional with a symmetric mountain pass geometry possesses an unbounded sequence of *nondegenerate critical values*. We believe that such a result is of independent interest to a broader class of variational problems. There is much room to select a function G which satisfies the assumption (G). Just as an example, we choose

$$G(x,t,\xi) = |t|^r |\xi|^q + \int_0^t g(x,s)\, ds,$$

where $r, q \geq 2$ and g satisfies the following assumption

- (g_1) $g \in C(\mathbb{R}^N, \mathbb{R})$, $g(x,t) = g(|x|,t)$, and $g(x,t)/t$ is bounded on $\mathbb{R}^N \times ([-R,R]\backslash\{0\})$ for any $R > 0$.

Then we have the following corollary of Theorem 1.3.

Corollary 1.4. *Let $N \geq 2$, $r \geq 2$ and $q \geq 2$. Assume (V), (F_1)–(F_5) and (g_1). Then for any $m \in \mathbb{N}$ there exists $\varepsilon_m > 0$ such that the quasilinear Schrödinger equation*

$$-\Delta u + \varepsilon q|u|^r \Delta_q u + \varepsilon(q-1)r|u|^{r-2}u|\nabla u|^q + V(x)u = f(x,u) + \varepsilon g(x,u) \quad \text{in } \mathbb{R}^N \tag{1.9}$$

possesses at least m distinct radial solutions provided that $|\varepsilon| \leq \varepsilon_m$, where $\Delta_q u = \operatorname{div}(|\nabla u|^{q-2}\nabla u)$ is the q-Laplacian of u.

Proof. It can be verified that the function G selected above satisfies (G) when $r \geq 2$ and $q \geq 2$ and g satisfies (g_1). In addition, for this particular G, (1.3) is just (1.9). We obtain the result by Theorem 1.3. □

We conjecture that most of the radial solutions obtained in Theorems 1.1 and 1.3 are sign-changing solutions. A clue is provided by the well known result that for $2 < p < 2^*$ the equation

$$-\Delta u + u = |u|^{p-2}u \ \text{ in } \ \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N),$$

which is a special case of the equations considered in those two theorems, has only one positive radial solution.

We turn to consider (1.4), which is a perturbation of the Choquard equation

$$-\Delta u + \omega u - \phi_u u = 0 \quad \text{in } \mathbb{R}^3, \quad u \in H^1(\mathbb{R}^3). \tag{1.10}$$

Choquard equations including the form of (1.10) have been well studied; see the survey paper [23] by Moroz and Van Schaftingen. We assume

(g_2) $g \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$, $g(x,t) = g(|x|,t)$, $\lim_{t\to 0} g(x,t)/t = 0$ uniformly for $x \in \mathbb{R}^3$, and $g(x,t)$ is bounded on $\mathbb{R}^3 \times [-R, R]$ for any $R > 0$.

Our third main result is for (1.4) which is as follows.

Theorem 1.5. *Suppose that (g_2) holds. Then for any given $m \in \mathbb{N}$ there exists $\varepsilon_m > 0$ such that (1.4) has at least m distinct radial solutions provided that $|\varepsilon| \leq \varepsilon_m$.*

In (1.4), let $\lambda = \frac{1}{\varepsilon^2}$ and replace u with u/ε. Then (1.4) is converted to the equation

$$\begin{cases} -\Delta u + \omega u - \lambda \phi_u u + g(x,u) = 0 \quad \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3). \end{cases}$$

In view of the fact that $\phi_u \in D^{1,2}(\mathbb{R}^3)$ is the unique weak solution of the equation $-\Delta \phi = u^2$ in $\mathbb{R}^3$, the above Choquard equation is equivalent to the Schrödinger-Poisson system

$$\begin{cases} -\Delta u + \omega u - \lambda \phi u + g(x,u) = 0 \quad \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 \quad \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \quad \phi \in D^{1,2}(\mathbb{R}^3). \end{cases} \tag{1.11}$$

This yields the following corollary of Theorem 1.5. Indeed, the conclusion of the corollary corresponds to the $0 < \varepsilon < \varepsilon_m$ part of that of Theorem 1.5.

Corollary 1.6. *Suppose that (g_2) holds. Then for any given $m \in \mathbb{N}$ there exists $\Lambda_m > 0$ such that, for $\lambda \geq \Lambda_m$, (1.11) has at least m distinct radial solutions (u, ϕ).*

Schrödinger-Poisson systems as in (1.11) have been extensively studied, especially in the case $\lambda < 0$. Here we have $\lambda > 0$, a case which is much less studied. In the latter case, Mugnai in [24] proved that for fixed $\omega > 0$ there exist infinitely many triples $(\lambda, u, \phi) \in \mathbb{R}^+ \times H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ which solve system (1.11), assuming that $g(x,t)$ is odd in t and satisfies some conditions which are much stronger than our condition (g_2). Here λ is a Lagrange multiplier and is part of the solutions obtained. For fixed $\omega, \lambda > 0$, a radial solution (u, ϕ) was obtained in [24] under even stronger assumptions. Jeong and Seok in [13] proved existence of a radial solution for fixed $\omega > 0$ and for $\lambda > 0$ sufficiently large, assuming (g_2) together with the assumption that $\overline{\lim}_{|t|\to\infty} |g(x,t)|/|t|^p < \infty$ for some $p \in (1,5)$ uniformly for $x \in \mathbb{R}^3$, and thus relaxed the conditions in [24]. Corollary 1.6 tremendously improves the related results in [13, 24] mentioned above. Not only is assumption (g_2) much weaker than those in the related results in [13, 24], but also the conclusion of Corollary 1.6 is much stronger than those in [13, 24] since only one solution was obtained for fixed λ in [13, 24].

We shall use a unified idea to prove our main theorems by showing that the unperturbed energy functionals have an infinite sequence of *nondegenerate critical values*. The methods used in this paper are quite different from those in [1, 7, 13, 24, 26, 28, 31], which cannot be extended to yield our results. The obstacle is that one does not know whether critical values or critical points obtained via classical minimax procedure are nondegenerate or not, and they may not be inherited when the functional is slightly perturbed.

The concept of essential values was introduced by Degiovanni and Lancelotti [10] in the study of nonsmooth functionals with perturbed symmetry. The advantage of essential values is that they are inherited when the functional is slightly perturbed, but the disadvantage is that in general they are extremely difficult to construct. It turns out that essential values are critical values if the functional satisfies the (PS) condition. In this sense, essential values are nondegenerate critical values. A theory of essential values has been developed in [10] and we shall use this theory to prove our main results. In this paper, a real number c is called a dual essential value of a functional I if $-c$ is an essential value of $-I$.

Let us sketch the ideas of the proofs of Theorems 1.1, 1.3, and 1.5. For equation (1.2), we consider the unperturbed constrained functional $\Phi|_{\mathcal{M}}$ in the spirit of [7], where

$$\Phi(u) = \int_{\mathbb{R}^N} F(u)\, dx, \quad \mathcal{M} = \{u \in H^1_r(\mathbb{R}^N) : \ |\nabla u|_2 = 1\},$$

where $|\cdot|_r$ stands for the usual norm in $L^r(\mathbb{R}^N)$. The first step of the proof of Theorem 1.1 is to seek a sequence of positive dual essential values of the unperturbed functional $\Phi|_{\mathcal{M}}$ tending to 0. To this aim we first show that if the super-level set $(\Phi|_{\mathcal{M}})_a \neq \emptyset$ for some $a > 0$ then it can be deformed in $(\Phi|_{\mathcal{M}})_b$ for some $b \in (0, a)$ to a single point (see Lemma 3.1). This result is the core of our approach. Since f is very general, the construction of this deformation is quite far from being trivial. Using this deformation result and the asymptotic properties of the minimax values introduced by Berestycki and P.-L. Lions [7], we can detect the topological change of super-level sets of Φ, and then establish the existence of a sequence of positive dual essential values tending to 0 (see Lemma 3.2). In the second step, we introduce and study for ε and $R > 1$ a class of modified functionals $\Phi_{\varepsilon,R}|_{\mathcal{M}}$ given by

$$\Phi_{\varepsilon,R}(u) = \int_{\mathbb{R}^N} F(u)\,dx + \varepsilon\eta_R\big(|u|_2^2\big)\int_{\mathbb{R}^N} G(\eta_R(u)u)\,dx,$$

where $\eta_R \in C^\infty(\mathbb{R},[0,1])$, $\eta(t) = 1$ for $|t| \le R-1$, $\eta(t) = 0$ for $|t| \ge R$, and $G(t) = \int_0^t g(s)ds$. Note that in the modification not only is G truncated so that the integral $\int_{\mathbb{R}^N} G(\eta_R(u)u)\,dx$ is well defined but an additional factor $\eta_R\big(|u|_2^2\big)$ is attached as well. The aim of adding this factor is to ensure that the perturbation term is uniformly small when $|\varepsilon|$ is small. Then by the essential value theory, we can find as many dual essential values of $\Phi_{\varepsilon,R}|_{\mathcal{M}}$ as prescribed for $|\varepsilon|$ suitably small such that the distance between two adjacent dual essential values has a positive lower bound independent of R (see Lemma 3.4). By verifying the (PS) condition, we see that the dual essential values are critical values of $\Phi_{\varepsilon,R}|_{\mathcal{M}}$. Finally, through a proper scale change and a uniform L^∞ estimate, by selecting and fixing an R large enough we derive as many solutions of the original equation as prescribed when $|\varepsilon|$ is sufficiently small (see Section 3.3), concluding the proof of Theorem 1.1.

While the main idea of the proofs of Theorems 1.1 and 1.3 is the same, the detailed techniques between the two proofs are quite different. To obtain Theorem 1.3, we shall first prove that under the hypotheses of the symmetric mountain pass theorem of Ambrosetti and Rabinowitz an even functional has a sequence of essential values tending to $+\infty$ (see Theorem 4.1). This result reinforces the conclusion of the theorem of Ambrosetti and Rabinowitz and is of independent interest. It is more convenient to be applied to problems with perturbed symmetry. Using this result, we can find an unbounded sequence of positive essential values of the unperturbed functional $J := J_0$. In the second step, we introduce a new class of functionals $J_{\varepsilon,R,\theta} : H_r^1(\mathbb{R}^N) \to \mathbb{R}$ to modify J_ε:

$$J_{\varepsilon,R,\theta}(u) = J(u) - \varepsilon\eta_R\big(|u|_{2+2\theta}^{2+2\theta}\big)\int_{\mathbb{R}^N} G\Big(x,\,\eta_R\big(|t|^{1+\theta}\big)|t|^\theta t,\,\eta_R\big(|\xi|^2\big)\xi\Big)\,dx,$$

where $\theta \in (0, 2/(N-2))$ and η_R $(R > 1)$ is as above. The new parameter θ introduced in the definition of $J_{\varepsilon,R,\theta}$ will paly a key role in the verification of the (PS) condition via the compactness of the Sobolev embedding $H_r^1(\mathbb{R}^N) \hookrightarrow L^{2+2\theta}(\mathbb{R}^N)$ (see Lemma 4.5). The functional $J_{\varepsilon,R,\theta}$ is smooth and $\sup_E |J_{\varepsilon,R,\theta} - J| \le C_R|\varepsilon|$. In view of the nondegeneracy nature of essential values of J, we see that the number of critical values of $J_{\varepsilon,R,\theta}$ tends to infinity uniformly in θ and R as $\varepsilon \to 0$. More precisely, given m, $J_{\varepsilon,R,\theta}$ with $|\varepsilon|$ small has m critical points $\{u_{\varepsilon,R,\theta,j}\}_{j=1}^m$ so that the associated critical values $c_{\varepsilon,R,\theta,j} = J_{\varepsilon,R,\theta}(u_{\varepsilon,R,\theta,j})$ satisfies $0 < c_0^{-1} \le c_{\varepsilon,R,\theta,j} \le c_0$ and $c_{\varepsilon,R,\theta,j+1} - c_{\varepsilon,R,\theta,j} \ge 1$ for some positive constant c_0 independent of ε, R, θ, and j (see Proposition 4.6). In the third step, we first prove that $u_{\varepsilon,R,\theta,j}$ $(1 \le j \le m)$ are bounded in the Sobolev norm and then use a delicate estimate to show that $|u_{\varepsilon,R,\theta,j}|_\infty + |\nabla u_{\varepsilon,R,\theta,j}|_\infty \le C$ for some C independent of ε, R, θ, and j (see Lemma 4.10). This implies that, for R sufficiently large, $u_{\varepsilon,\theta,j} := u_{\varepsilon,R,\theta,j}$ $(1 \le j \le m)$ are critical points of

$$J_{\varepsilon,\theta}(u) = J(u) - \varepsilon \int_{\mathbb{R}^N} G(x,\, |t|^\theta t,\, \xi) dx.$$

In the last step, we choose a sequence $\{\theta_k\}_{k=1}^\infty \subset (0, 2/(N-2))$ such that $\theta_k \to 0^+$ and prove that $\nabla u_{\varepsilon,\theta_k,j} \to \nabla u_{\varepsilon,j}$ a.e. in $\mathbb{R}^N$ as $k \to \infty$, where $u_{\varepsilon,j}$ is the weak limit of $u_{\varepsilon,\theta_k,j}$ in the Sobolev space as $k \to \infty$ (see Lemma 4.11). Then we show that $u_{\varepsilon,j}$ $(1 \le j \le m)$ are the strong limits of $u_{\varepsilon,\theta_k,j}$ $(1 \le j \le m)$ and thus are solutions of the original equation (1.3). That they are different from each other is a consequence of the energy estimate given in Proposition 4.6.

The proof of Theorem 1.5 shares a similar approach with the proof of Theorem 1.3. However, the modification of the original functional is more complicated, and we introduce and consider the following class of functionals

$$\begin{aligned} J_{\varepsilon,R,\theta}(u) = \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2\, dx \\ + \frac{1}{\varepsilon^2}\eta_R\big(|u|_{2+2\theta}^{2+2\theta} + |u|_{p(1+\theta)}^{p(1+\theta)}\big) \int_{\mathbb{R}^3} G\big(x, \varepsilon\eta_R(|u|^{1+\theta})|u|^\theta u\big)\, dx, \end{aligned}$$

where $G(x,t) = \int_0^t g(x,s)\, ds$, $2 < p < 6$ is a fixed number and $0 < \theta < (6-p)/p$ is arbitrary but fixed. Here we add the term $|u|_{p(1+\theta)}^{p(1+\theta)}$ in order to accommodate $J_{\varepsilon,R,\theta}$ with an estimate on $\sup_{H_r^1(\mathbb{R}^3)} |J_{\varepsilon,R,\theta} - J|$ as well as the (PS) condition under assumption (g_2) (see Lemmas 5.5 and 5.6). Then we can argue as in the proof of Theorem 1.3 to prove Theorem 1.5.

The rest of paper is organized as follows. We shall recall some basic facts from [10] in Section 2 and prove Theorems 1.1, 1.3, and 1.5 in Sections 3, 4, and 5, respectively. We shall give more results which are variants of those above in Section 6.

Notation. Throughout this paper we shall make use of the following notation:

- $H_r^1(\mathbb{R}^N) := \{u \in H^1(\mathbb{R}^N) \,|\, u(x) = u(|x|)\}$.
- The norm in $L^q(\mathbb{R}^N)$ is denoted by $|\cdot|_q$, where $1 \le q \le \infty$.
- $\mathbb{S}^{k-1}$ is the unit sphere in $\mathbb{R}^k$.
- $B_r(x)$ stands for a ball with radius r and center x in various space. We use B_r to denote $B_r(0)$.
- $o(1)$ means a quantity tending to 0.
- C, $C(\cdot)$, C_j and C_R stand for various positive constants.

2 Preliminaries

To prove our main results, we shall use the essential value theory developed by Degiovanni and Lancelotti in [10]. Let us recall some concepts and facts from [10] which will be used in the following sections and which we shall not state as generally as in [10]. Let E be a Banach space (or a complete $C^{1,1}$ Finsler manifold) and $I \in C^1(E, \mathbb{R})$. For $c \in \bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$, set $I^c = \{u \in E \,|\, I(u) \le c\}$ for the sub-level set.

Definition 2.1. *Let α, $\beta \in \bar{\mathbb{R}}$ with $\alpha \le \beta$. The pair (I^β, I^α) is said to be trivial for I, if for every neighborhood $[\alpha', \alpha'']$ of α and $[\beta', \beta'']$ of β, where $\alpha', \alpha'', \beta', \beta'' \in \bar{\mathbb{R}}$, there exist two closed subsets A, B of E such that $I^{\alpha'} \subseteq A \subseteq I^{\alpha''}$, $I^{\beta'} \subseteq B \subseteq I^{\beta''}$ and such that A is a strong deformation retract of B.*

Definition 2.2. *A number $c \in \mathbb{R}$ is said to be an essential value of I, if for every $\varepsilon > 0$ there exist two numbers α, $\beta \in (c - \varepsilon, c + \varepsilon)$ with $\alpha < \beta$ such that the pair (I^β, I^α) is not trivial for I.*

Lemma 2.3 (Degiovanni and Lancelotti [10])**.** *Let α, $\beta \in \bar{\mathbb{R}}$ with $\alpha < \beta$. If I has no essential value in (α, β) then the pair (I^β, I^α) is trivial for I.*

Lemma 2.4 (Degiovanni and Lancelotti [10])**.** *Let $c \in \mathbb{R}$ be an essential value of I. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that every functional $J \in C^1(E, \mathbb{R})$ with $\sup_{u \in E} |J(u) - I(u)| < \delta$ admits an essential value in $(c - \varepsilon, c + \varepsilon)$.*

Lemma 2.5 (Degiovanni and Lancelotti [10])**.** *Let $c \in \mathbb{R}$ be an essential value of I. If I satisfies $(PS)_c$ condition then c is a critical value of I.*

To facilitate our arguments in Section 3, we give a dual version of the above definitions and facts. For $c \in \bar{\mathbb{R}}$, set $I_c = \{u \in E \,|\, I(u) \ge c\}$ for the super-level set.

Definition 2.1′. *Let α, $\beta \in \bar{\mathbb{R}}$ with $\alpha \le \beta$. The pair (I_α, I_β) is said to be trivial for I if $((-I)^{-\alpha}, (-I)^{-\beta})$ is trivial for $-I$ in the sense of Definition 2.1.*

Definition 2.2′. *A number $c \in \mathbb{R}$ is said to be a dual essential value of I if $-c$ is an essential value of $-I$.*

Lemma 2.3′. *Let α, $\beta \in \bar{\mathbb{R}}$ with $\alpha < \beta$. If I has no dual essential value in (α, β), then the pair (I_α, I_β) is trivial for I.*

Lemma 2.4′. *Let $c \in \mathbb{R}$ be a dual essential value of I. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that every functional $J \in C^1(E, \mathbb{R})$ with $\sup_{u \in E} |J(u) - I(u)| < \delta$ admits a dual essential value in $(c - \varepsilon, c + \varepsilon)$.*

Lemma 2.5′. *Let $c \in \mathbb{R}$ be a dual essential value of I. If I satisfies $(PS)_c$ condition, then c is a critical value of I.*

The proofs of Lemmas 2.3′–2.5′ can be conducted exactly the same way as in [10]. They can also be deduced as consequences from Lemmas 2.3–2.5, considering $-I$ in place of I.

Lemma 2.4 claims that essential values remain if the functional is slightly perturbed. Critical values of a functional do not have such a property. This makes essential value theory a powerful tool in the study of perturbation problems. In applications, however, to construct essential values is usually more difficult than to construct critical values.

3 Proof of Theorem 1.1

In this section we deal with the scalar field equation (1.2) and prove Theorem 1.1. We first remark that, under assumption (f_3), we can make use of its stronger version

(f_3') $\lim_{t\to+\infty} f(t)/t^{2^*-1} = 0$

without any loss of generality; see [6, Pages 323–324] for details. Using the principle of symmetric criticality [25], we shall work in $H_r^1(\mathbb{R}^N)$ consisting of radial functions from $H^1(\mathbb{R}^N)$ and use its usual norm in this section. Set

$$\mathcal{M} = \left\{ u \in H_r^1(\mathbb{R}^N) \,\middle|\, \int_{\mathbb{R}^N} |\nabla u|^2 \, dx = 1 \right\}.$$

Clearly $\mathcal{M}$ is a complete smooth submanifold of $H_r^1(\mathbb{R}^N)$. Define $\Phi : H_r^1(\mathbb{R}^N) \to \mathbb{R}$ by

$$\Phi(u) = \int_{\mathbb{R}^N} F(u) \, dx.$$

Then $\Phi \in C^1(H_r^1(\mathbb{R}^N), \mathbb{R})$. We shall consider the constrained functional $\Phi|_{\mathcal{M}}$ and for $a > 0$ the super-level sets $(\Phi|_{\mathcal{M}})_a$. For simplicity of notation, we denote $\Phi_a = (\Phi|_{\mathcal{M}})_a$ and thus $\Phi_a = \{u \in \mathcal{M} \mid \Phi(u) \ge a\}$.

3.1 Essential values of $\Phi|_{\mathcal{M}}$

In this subsection we shall prove that $\Phi|_{\mathcal{M}}$ has a sequence of positive dual essential values tending to 0. To do that we first prove the following result which asserts that for any $a > 0$, if the super-level set $\Phi_a \neq \emptyset$ then there exists $b \in (0, a)$ such that Φ_a is contractible in Φ_b.

Let us first sketch the idea how the contraction mapping is constructed. We split up the construction into four steps. In the first step, we select $0 < R_1 < R_2$ large enough and deform the outer part $\{u(x) \mid |x| \ge R_2\}$ of $u \in \Phi_a$ to 0 while keeping the inner part $\{u(x) \mid |x| \le R_1\}$ of u unchanged. The second step is a process of growing a small tail $u_0(|x|)$ on the outer part $\{u(x) \mid |x| \ge R_2\}$ of each function u obtained after the first step, where u_0 is a smooth function such that $\mathrm{supp}(u_0) \subset (R_2, R_2 + 3\delta)$, $0 \le u_0 \le \zeta$, and $u_0(r) = \zeta$ for all $r \in (R_2 + \delta, R_2 + 2\delta)$, with $\delta > 0$ chosen sufficiently small. In the third step, we enlarge the flat part of the tail from the annular $B_{R_2+2\delta} \setminus B_{R_2+\delta}$ to a larger one $B_{R_3} \setminus B_{R_2+\delta}$, where R_3 is sufficiently large so that the outer part of the functions obtained after the third step becomes the dominant part. Then in the fourth step, we deform the inner part $\{u(x) \,|\, |x| \le R_2\}$ to 0 for u obtained after the third step. In the whole process of the deformation we manage to have that the value of Φ is always larger than $\frac{a}{2}$ and the $D^{1,2}(\mathbb{R}^N)$ norm remains bounded. We then use dilation to define the desired deformation from $[0,1] \times \Phi_a$ to $\mathcal{M}$.

Lemma 3.1. *Let $a > 0$ be such that $\Phi_a \neq \emptyset$. Then there exist $u^* \in \mathcal{M}$ and a map $h \in C([0,1] \times \Phi_a, \mathcal{M})$ such that*

(i) *$h(0,u) = u$ and $h(1,u) \equiv u^*$ for any $u \in \Phi_a$;*

(ii) *$b := \inf_{(s,u)\in[0,1]\times\Phi_a} \Phi(h(s,u)) > 0$.*

Proof. By (f_2), there exists $t_0 > 0$ such that $f(t)/t < 0$ if $0 < |t| \le t_0$. Then F is increasing on $[-t_0, 0]$ and decreasing on $[0, t_0]$. By the Strauss inequality ([6, Radial Lemma A.III]), one can choose $R_1 > 0$ such that $|u(x)| \le t_0$ for any $u \in \Phi_a$ and $x \in \mathbb{R}^N \setminus B_{R_1}$. We fix $R_2 > R_1$ and choose a C^∞ function $\chi : [0, +\infty) \to [0,1]$ satisfying $\chi(r) = 1$ if $0 \le r \le R_1$ and $\chi(r) = 0$ if $r \ge R_2$. Define a map $h_1 \in C([0,1] \times \Phi_a, H^1_r(\mathbb{R}^N))$ by

$$h_1(s,u)(x) = (1 - s + s\chi(|x|))u(x).$$

Then, for $(s,u) \in [0,1] \times \Phi_a$, since F is increasing in $[-t_0, 0]$ and decreasing in $[0, t_0]$, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} F(h_1(s,u))\,dx &= \int_{B_{R_1}} F(u)\,dx + \int_{\mathbb{R}^N\setminus B_{R_1}} F((1-s+s\chi(|x|))u)\,dx \\
&= \int_{\mathbb{R}^N} F(u)\,dx + \int_{\mathbb{R}^N\setminus B_{R_1}} \big(F((1-s+s\chi(|x|))u) - F(u)\big)\,dx \\
&\ge \int_{\mathbb{R}^N} F(u)\,dx \ge a. \qquad (3.1)
\end{aligned}$$

Using (f_2), (f_3) and the Sobolev inequality we see that, for $(s,u) \in [0,1] \times \Phi_a$,

$$\begin{aligned}
\left|\int_{\mathbb{R}^N} F(s\chi(|x|)u)\,dx\right| &= \left|\int_{B_{R_2}} F(s\chi(|x|)u)\,dx\right| \le C_1 \int_{B_{R_2}} \left(u^2 + |u|^{2^*}\right) dx \\
&\le C_2(N) \left[\left(\int_{\mathbb{R}^N} |u|^{2^*}\,dx\right)^{\frac{2}{2^*}} + \int_{\mathbb{R}^N} |u|^{2^*}\,dx\right] \le C_3(N).
\end{aligned}$$

Setting $\mathcal{D}_1 = h_1(1, \Phi_a)$, then

$$\sup_{(s,u)\in[0,1]\times\mathcal{D}_1} \left|\int_{\mathbb{R}^N} F(su)\,dx\right| = \sup_{(s,u)\in[0,1]\times\Phi_a} \left|\int_{\mathbb{R}^N} F(s\chi(|x|)u)\,dx\right| \le C_3(N). \qquad (3.2)$$

Recall $F(\zeta) > 0$ by (f_4). Fix $R_3 > R_2 + 1$ such that

$$F(\zeta) \cdot \text{meas}\left(B_{R_3} \setminus B_{R_2+1}\right) > C_3(N) + a. \tag{3.3}$$

Set $\nu = -\min_{t\in[0,\zeta]} F(t) > 0$ and then choose $\delta \in \left(0, \frac{1}{3}\right)$ such that

$$2\nu\left[\text{meas}\left(B_{R_2+3\delta} \setminus B_{R_2}\right) + \text{meas}\left(B_{R_3+\delta} \setminus B_{R_3}\right)\right] < a. \tag{3.4}$$

Let $u_0 : [0, +\infty) \to [0, \zeta]$ be a C^∞ function having the properties

$$u_0(r) = \begin{cases} 0, & \text{if } 0 \le r \le R_2 \text{ or } r \ge R_2 + 3\delta, \\ \zeta, & \text{if } R_2 + \delta \le r \le R_2 + 2\delta. \end{cases}$$

Define a map $h_2 \in C([0,1] \times \mathcal{D}_1, H^1_r(\mathbb{R}^N))$ by

$$h_2(s,u)(x) = \begin{cases} u(x), & \text{if } x \in B_{R_2}, \\ su_0(|x|), & \text{if } x \in \mathbb{R}^N \setminus B_{R_2}. \end{cases}$$

Then, for $(s,u) \in [0,1] \times \mathcal{D}_1$, we deduce from (3.1) and (3.4) that

$$\begin{aligned} \int_{\mathbb{R}^N} F(h_2(s,u))\,dx &= \int_{B_{R_2}} F(u)\,dx + \int_{\mathbb{R}^N\setminus B_{R_2}} F(su_0(|x|))\,dx \\ &= \int_{\mathbb{R}^N} F(u)\,dx + \int_{\mathbb{R}^N\setminus B_{R_2}} F(su_0(|x|))\,dx \\ &\ge a - \nu \cdot \text{meas}\left(B_{R_2+3\delta} \setminus B_{R_2}\right) > \frac{a}{2}. \end{aligned} \tag{3.5}$$

Set $\mathcal{D}_2 = h_2(1, \mathcal{D}_1)$ and define a map $h_3 \in C([0,1] \times \mathcal{D}_2, H^1_r(\mathbb{R}^N))$ by

$$h_3(s,u)(x) = \begin{cases} u(x), & \text{if } x \in B_{R_2+\delta}, \\ \zeta, & \text{if } x \in B_{R_2+2\delta+s(R_3-R_2-2\delta)} \setminus B_{R_2+\delta}, \\ u_0(|x| - s(R_3 - R_2 - 2\delta)), & \text{if } x \in \mathbb{R}^N \setminus B_{R_2+2\delta+s(R_3-R_2-2\delta)}. \end{cases}$$

Then, for $(s,u) \in [0,1] \times \mathcal{D}_2$, we use (3.1) and (3.4) again to obtain

$$\begin{aligned} &\int_{\mathbb{R}^N} F(h_3(s,u))\,dx \\ &= \int_{B_{R_2}} F(u)\,dx + \int_{B_{R_2+\delta}\setminus B_{R_2}} F(u)\,dx + \int_{B_{R_2+2\delta+s(R_3-R_2-2\delta)}\setminus B_{R_2+\delta}} F(\zeta)\,dx \\ &\quad + \int_{B_{R_2+3\delta+s(R_3-R_2-2\delta)}\setminus B_{R_2+2\delta+s(R_3-R_2-2\delta)}} F(u_0(|x| - s(R_3 - R_2 - 2\delta)))\,dx \\ &\ge a - \nu \cdot \text{meas}\left(B_{R_2+\delta} \setminus B_{R_2}\right) \\ &\quad + F(\zeta) \cdot \text{meas}\left(B_{R_2+2\delta+s(R_3-R_2-2\delta)} \setminus B_{R_2+\delta}\right) \\ &\quad - \nu \cdot \text{meas}\left(B_{R_2+3\delta+s(R_3-R_2-2\delta)} \setminus B_{R_2+2\delta+s(R_3-R_2-2\delta)}\right) \\ &\ge a - \nu \cdot \text{meas}\left(B_{R_2+\delta} \setminus B_{R_2}\right) - \nu \cdot \text{meas}\left(B_{R_3+\delta} \setminus B_{R_3}\right) > \frac{a}{2}. \end{aligned} \tag{3.6}$$

Set $\mathcal{D}_3 = h_3(1, \mathcal{D}_2)$. For $u \in \mathcal{D}_3$ and $|x| \geq R_2$, we have $u(x) = \bar{u}(x)$, where

$$\bar{u}(x) = \begin{cases} u_0(|x|), & \text{if } x \in B_{R_2+\delta}, \\ \zeta, & \text{if } x \in B_{R_3} \setminus B_{R_2+\delta}, \\ u_0(|x| - (R_3 - R_2 - 2\delta)), & \text{if } x \in \mathbb{R}^N \setminus B_{R_3}. \end{cases}$$

This together with (3.3) and (3.4) implies

$$\begin{aligned} \int_{\mathbb{R}^N \setminus B_{R_2}} F(u)\, dx \geq{}& F(\zeta) \cdot \text{meas}\left(B_{R_3} \setminus B_{R_2+\delta}\right) \\ & - \nu \cdot \text{meas}\left(B_{R_2+\delta} \setminus B_{R_2}\right) - \nu \cdot \text{meas}\left(B_{R_3+\delta} \setminus B_{R_3}\right) \\ \geq{}& C_3(N) + a - \frac{a}{2} = C_3(N) + \frac{a}{2}. \end{aligned} \tag{3.7}$$

Define a map $h_4 \in C([0,1] \times \mathcal{D}_3, H_r^1(\mathbb{R}^N))$ by

$$h_4(s,u)(x) = \begin{cases} (1-s)u(x), & \text{if } x \in B_{R_2}, \\ u(x), & \text{if } x \in \mathbb{R}^N \setminus B_{R_2}. \end{cases}$$

Then, for $(s,u) \in [0,1] \times \mathcal{D}_3$, we see from (3.2) and (3.7) that

$$\begin{aligned} \int_{\mathbb{R}^N} F(h_4(s,u))\, dx &= \int_{B_{R_2}} F((1-s)u)\, dx + \int_{\mathbb{R}^N \setminus B_{R_2}} F(u)\, dx \\ &\geq -C_3(N) + C_3(N) + \frac{a}{2} = \frac{a}{2} > 0. \end{aligned} \tag{3.8}$$

We point out that $h_4(1,u) \equiv \bar{u}$ for $u \in \mathcal{D}_3$.

Now we define a map $\bar{h} \in C([0,1] \times \Phi_a, H_r^1(\mathbb{R}^N))$ by

$$\bar{h}(s,u) = \begin{cases} h_1(4s, u), & \text{if } 0 \leq s \leq \frac{1}{4}, \\ h_2(4s-1, h_1(1,u)), & \text{if } \frac{1}{4} \leq s \leq \frac{1}{2}, \\ h_3(4s-2, h_2(1, h_1(1,u))), & \text{if } \frac{1}{2} \leq s \leq \frac{3}{4}, \\ h_4(4s-3, h_3(1, h_2(1, h_1(1,u)))), & \text{if } \frac{3}{4} \leq s \leq 1. \end{cases}$$

It is clear that, for all $u \in \Phi_a$,

$$\bar{h}(0,u) = u, \quad \bar{h}(1,u) = \bar{u}.$$

From (3.1), (3.5), (3.6), and (3.8) we have

$$\inf_{(s,u) \in [0,1] \times \Phi_a} \int_{\mathbb{R}^N} F(\bar{h}(s,u))\, dx \geq \frac{a}{2}. \tag{3.9}$$

In addition, checking the process of the definition of $\bar{h}$ we see that

$$\sup_{(s,u) \in [0,1] \times \Phi_a} \int_{\mathbb{R}^N} |\nabla(\bar{h}(s,u))|^2\, dx < +\infty. \tag{3.10}$$

Finally we define the map $h \in C([0,1] \times \Phi_a, \mathcal{M})$ by

$$[h(s,u)](x) = [\bar{h}(s,u)](Tx),$$

where

$$T = T(s,u) := \left(\int_{\mathbb{R}^N} |\nabla(\bar{h}(s,u))|^2 \, dx \right)^{\frac{1}{N-2}}.$$

Then $h(0,u) = u$, $h(1,u) \equiv u^*$ for all $u \in \Phi_a$ and some $u^* \in \mathcal{M}$. Moreover, for $(s,u) \in [0,1] \times \Phi_a$,

$$\begin{aligned} \Phi(h(s,u)) &= \int_{\mathbb{R}^N} F(h(s,u)) \, dx = T^{-N} \int_{\mathbb{R}^N} F(\bar{h}(s,u)) \, dx \\ &= \left(\int_{\mathbb{R}^N} |\nabla(\bar{h}(s,u))|^2 \, dx \right)^{-\frac{N}{N-2}} \int_{\mathbb{R}^N} F(\bar{h}(s,u)) \, dx. \end{aligned}$$

We see from (3.9) and (3.10) that

$$\inf_{(s,u) \in [0,1] \times \Phi_a} \Phi(h(s,u)) > 0,$$

concluding the proof. □

Using Lemma 2.3′ and Lemma 3.1, we show that $\Phi|_{\mathcal{M}}$ has a sequence of positive dual essential values tending to 0. Such a result strengthens the related one in [7] since essential values are nondegenerate critical values.

Lemma 3.2. *Set*

$$\Lambda = \{c > 0 \,|\, c \text{ is a dual essential value of } \Phi|_{\mathcal{M}}\}.$$

Then $\Lambda \neq \emptyset$ *and* $\inf \Lambda = 0$.

Proof. Define as in [7],

$$\beta_k = \sup_{A \in \Gamma_k} \inf_{u \in A} \Phi(u),$$

where $\Gamma_k = \{A \in \Sigma(\mathcal{M}) \,|\, \gamma(A) \geq k\}$, $\Sigma(\mathcal{M})$ denotes the set of compact and symmetric subsets of $\mathcal{M}$ and γ stands for the Krasnoselskii genus. According to [7, Proposition 2], $\beta_k > 0$ and $\lim_{k\to\infty} \beta_k = 0$. Define

$$b_k = \sup_{\varphi \in \Gamma_k^*} \inf_{\sigma \in \mathbb{S}^{k-1}} \Phi(\varphi(\sigma)),$$

where $\Gamma_k^* = \{\varphi \in C(\mathbb{S}^{k-1}, \mathcal{M}) \,|\, \varphi \text{ is odd}\}$. By the proof of [7, Proposition 2] we see that $b_k > 0$ for all $k \in \mathbb{N}$. Since $\varphi(\mathbb{S}^{k-1}) \in \Gamma_k$ for $\varphi \in \Gamma_k^*$, we have $b_k \leq \beta_k$. Therefore

$$\lim_{k\to\infty} b_k = 0.$$

Clearly, $b_1 = \sup_{u\in\mathcal{M}} \Phi(u)$ is a dual essential value of $\Phi|_{\mathcal{M}}$. Hence $\Lambda \neq \emptyset$. It remains to prove that $\mu := \inf \Lambda = 0$. If this were false, then $\mu > 0$ and there would exist $k \in \mathbb{N}$ such that $0 < b_{k+1} < b_k < \mu$. We fix three numbers β', β, $\beta'' \in \mathbb{R}$ such that $0 < b_{k+1} < \beta' < \beta < \beta'' < b_k < \mu$ and choose $\varphi \in \Gamma_k^*$ satisfying

$$\inf_{u\in\varphi(\mathbb{S}^{k-1})} \Phi(u) \geq \beta''. \tag{3.11}$$

Applying Lemma 3.1, we obtain a map $h \in C([0,1] \times \varphi(\mathbb{S}^{k-1}), \mathcal{M})$ satisfying

(i) $h(0,u) = u$ and $h(1,u) \equiv u^* \in \mathcal{M}$ for any $u \in \varphi(\mathbb{S}^{k-1})$;

(ii) $\inf_{(s,u)\in[0,1]\times\varphi(\mathbb{S}^{k-1})} \Phi(h(s,u)) > 0.$

Let $\alpha'' \in \mathbb{R}$ be such that

$$0 < \alpha'' < \min\left\{ b_{k+1}, \inf_{(s,u)\in[0,1]\times\varphi(\mathbb{S}^{k-1})} \Phi(h(s,u)) \right\}. \tag{3.12}$$

By Lemma 2.3′, the pair (Φ_0, Φ_β) is trivial since $\Phi|_{\mathcal{M}}$ has no dual essential value in $(0,\beta)$. Then there exist two closed subsets A, B of $\mathcal{M}$ such that $\Phi_{\beta''} \subseteq B \subseteq \Phi_{\beta'}$, $\Phi_{\alpha''} \subseteq A$, and B is a strong deformation retract of A. In particular, there exists $\eta \in C(A,B)$ such that $\eta(u) = u$ for $u \in B$. Let us define a map $\tau : [0,1] \times \mathbb{S}^{k-1} \to \mathcal{M}$ by

$$\tau(s,\sigma) = \eta \circ h(s, \varphi(\sigma)).$$

That τ is well defined is a consequence of (3.12) and the fact that $\Phi_{\alpha''} \subseteq A$. Then τ is continuous, $\tau(1,\sigma) \equiv \eta(u^*) \in \mathcal{M}$ for any $\sigma \in \mathbb{S}^{k-1}$, and $\tau([0,1] \times \mathbb{S}^{k-1}) \subseteq \Phi_{\beta'}$ since $B \subseteq \Phi_{\beta'}$. Now we define a map $\psi : \mathbb{S}^k \to \mathcal{M}$ by

$$\psi(\sigma_1, \sigma_2, \cdots, \sigma_{k+1}) = \begin{cases} \tau\left(\sigma_{k+1}, \frac{1}{\sqrt{1-\sigma_{k+1}^2}}(\sigma_1, \sigma_2, \cdots, \sigma_k)\right), & \text{if } 0 \leq \sigma_{k+1} < 1, \\ \eta(u^*), & \text{if } \sigma_{k+1} = 1, \\ -\psi(-\sigma_1, -\sigma_2, \cdots, -\sigma_{k+1}), & \text{if } -1 \leq \sigma_{k+1} < 0. \end{cases}$$

By (3.11) and the fact that $\Phi_{\beta''} \subseteq B$, $\varphi(\mathbb{S}^{k-1}) \subseteq B$. Then for $\sigma \in \mathbb{S}^{k-1}$,

$$\tau(0,\sigma) = \eta \circ h(0, \varphi(\sigma)) = \eta \circ \varphi(\sigma) = \varphi(\sigma).$$

The fact that φ is odd and continuous on $\mathbb{S}^{k-1}$ implies that ψ is odd and continuous on $\mathbb{S}^k$. Then $\psi \in \Gamma_{k+1}^*$ and $\psi(\mathbb{S}^k) \subseteq \Phi_{\beta'}$ from which we have a contradiction, $b_{k+1} \geq \beta'$. ☐

3.2 Critical values of $\Phi_{\varepsilon,R}|_{\mathcal{M}}$

Set

$$G(t) = \int_0^t g(s)\, ds.$$

Note that in Theorem 1.1 the only control imposed on g is (1.6). To find solutions of (1.2) by variational methods, we need to truncate G so that the modified functional is well defined in $H^1_r(\mathbb{R}^N)$. For the moment, let $R>1$ be an arbitrary but fixed number. Choose $\eta_R \in C^\infty(\mathbb{R},[0,1])$ such that

$$|\eta_R'(t)| \le 2, \quad \eta_R(t) = 1 \text{ if } |t| \le R-1, \quad \operatorname{supp}(\eta_R) \subset (-R,R).$$

Define a functional $\Phi_{\varepsilon,R} : H^1_r(\mathbb{R}^N) \to \mathbb{R}$ by

$$\Phi_{\varepsilon,R}(u) = \int_{\mathbb{R}^N} F(u)\,dx + \varepsilon \eta_R\big(|u|_2^2\big) \int_{\mathbb{R}^N} G(\eta_R(u)u)\,dx.$$

Clearly, $\Phi_{\varepsilon,R} \in C^1(H^1_r(\mathbb{R}^N),\mathbb{R})$. Note that in the definition of $\Phi_{\varepsilon,R}$ not only is G truncated so that the integral $\int_{\mathbb{R}^N} G(\eta_R(u)u)\,dx$ is well defined but an additional factor $\eta_R\big(|u|_2^2\big)$ is added to this term as well. The aim of adding this factor is to guarantee the validity of the following result which is the key point to start with.

Lemma 3.3. *There exists $C_R > 0$ independent of ε such that*

$$\sup_{u\in\mathcal{M}} |\Phi_{\varepsilon,R}(u) - \Phi(u)| \le |\varepsilon| C_R.$$

The proof is easy and we drop it. Let $m \in \mathbb{N}$ be an arbitrary but fixed integer. By Lemmas 2.4′, 3.2 and 3.3, we have

Lemma 3.4. *There exist $c_0 \in (0,1)$ independent of R and $\varepsilon_1(R) > 0$ for $R>1$ such that if $|\varepsilon| \le \varepsilon_1(R)$ then $\Phi_{\varepsilon,R}|_{\mathcal{M}}$ has m dual essential values $\{c_{\varepsilon,R,j}\}_{j=1}^m$ satisfying*

$$0 < c_0 \le c_{\varepsilon,R,j} \le 1, \quad \forall\, 1 \le j \le m \tag{3.13}$$

and

$$c_{\varepsilon,R,j+1}^{-\frac{N-2}{2}} - c_{\varepsilon,R,j}^{-\frac{N-2}{2}} \ge N\left(\frac{2N}{N-2}\right)^{\frac{N-2}{2}}, \quad \forall\, 1 \le j \le m-1. \tag{3.14}$$

Lemma 3.5. *There exists $\varepsilon_2(R) \in (0,\varepsilon_1(R))$ for $R>1$ such that if $|\varepsilon| \le \varepsilon_2(R)$ then*

$$\Phi_{\varepsilon,R}(u) \le C - \frac{\ell}{4}\int_{\mathbb{R}^N} u^2\,dx, \quad \forall\, u \in \mathcal{M},$$

where $C>0$ is a constant independent of ε and R, and $\ell > 0$ is the number as in (f_2).

Proof. By (f_1)–(f_3), there exists $C>0$ such that

$$F(t) \le -\frac{\ell}{4}t^2 + C|t|^{2^*}, \quad \text{for } t \in \mathbb{R}.$$

Then, for $u \in \mathcal{M}$, we have

$$\Phi(u) = \int_{\mathbb{R}^N} F(u)\,dx \le -\frac{\ell}{4}\int_{\mathbb{R}^N} u^2\,dx + C\int_{\mathbb{R}^N} |u|^{2^*}\,dx \le -\frac{\ell}{4}\int_{\mathbb{R}^N} u^2\,dx + C.$$

Let C_R be as in Lemma 3.3 and fix a number $\varepsilon_2(R) \in (0,\min\{\varepsilon_1(R), C_R^{-1}\})$. Then the result follows from the above estimate and Lemma 3.3. □

The next result is [14, Lemma 2.12].

Lemma 3.6. *If $h \in C(\mathbb{R}, \mathbb{R})$ satisfies $h(t) \le 0$ for $|t|$ small and*

$$\lim_{t\to\infty} \frac{h(t)}{|t|^{2^*}} = 0,$$

then the functional $\int_{\mathbb{R}^N} h(u)\, dx$ is weakly sequentially upper semi-continuous in $H_r^1(\mathbb{R}^N)$.

Lemma 3.7. *Assume that $h \in C(\mathbb{R}, \mathbb{R})$ satisfies $h(t) \ge Ct^2$ for $|t|$ small and*

$$\lim_{t\to\infty} \frac{h(t)}{|t|^{2^*}} = 0. \tag{3.15}$$

If $u_n \to u$ weakly in $H_r^1(\mathbb{R}^N)$ and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} h(u_n)\, dx = \int_{\mathbb{R}^N} h(u)\, dx, \tag{3.16}$$

then $u_n \to u$ strongly in $L^2(\mathbb{R}^N)$.

Proof. We only need to prove that any subsequence of $\{u_n\}$ has a subsequence converging to u strongly in $L^2(\mathbb{R}^N)$. We keep the symbol $\{u_n\}$ for its subsequences and assume that $u_n \to u$ a.e. in $\mathbb{R}^N$. By the Strauss inequality, there exists $R_0 > 0$ such that

$$h(u_n(x)) \ge Cu_n^2(x) \ge 0 \quad \text{and} \quad h(u(x)) \ge Cu^2(x) \ge 0 \tag{3.17}$$

for all $n \in \mathbb{N}$ and $x \in \mathbb{R}^N \setminus B_{R_0}$. Using (3.15) and the Sobolev inequality, one can easily show that $h(u_n) \to h(u)$ strongly in $L^1(B_{R_0})$. Combining this with (3.16) yields that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N\setminus B_{R_0}} h(u_n)\, dx = \int_{\mathbb{R}^N\setminus B_{R_0}} h(u)\, dx.$$

Since by (3.17) $h(u_n(x)) \ge 0$ and $h(u(x)) \ge 0$ if $x \in \mathbb{R}^N \setminus B_{R_0}$ and since $u_n(x) \to u(x)$ a.e., it follows that $h(u_n) \to h(u)$ strongly in $L^1(\mathbb{R}^N \setminus B_{R_0})$. Then using again (3.17) and the a.e. convergence of u_n to u one can deduce that $u_n \to u$ strongly in $L^2(\mathbb{R}^N \setminus B_{R_0})$. Since we also have $u_n \to u$ strongly in $L^2(B_{R_0})$, the result follows. □

Lemma 3.8. *There exists $R_0 > 1$ such that for $R > R_0$ there exists $\varepsilon_3(R) \in (0, \varepsilon_2(R))$ such that if $|\varepsilon| \le \varepsilon_3(R)$ then $\Phi_{\varepsilon,R}|_{\mathcal{M}}$ satisfies the $(PS)_c$ condition for $c > 0$.*

Proof. Let $|\varepsilon| \le \varepsilon_2(R)$ and assume $\{u_n\} \subset \mathcal{M}$ to be a $(PS)_c$ sequence with $c > 0$, that is,

$$\Phi_{\varepsilon,R}(u_n) \to c \quad \text{and} \quad \|(\Phi_{\varepsilon,R}|_{\mathcal{M}})'(u_n)\| \to 0, \quad \text{as } n \to \infty.$$

By Lemma 3.5, there exists $C_1 > 0$ such that

$$\int_{\mathbb{R}^N} u_n^2\, dx \le C_1, \quad \text{for all } n \in \mathbb{N}.$$

Then $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^N)$ and, extracting a subsequence, one can assume that $u_n \to u$ weakly in $H_r^1(\mathbb{R}^N)$.

Set $R_0 = C_1 + 1$ and fix $R > R_0$. We have $\eta_R(|u_n|_2^2) = 1$. Thus

$$\Phi_{\varepsilon,R}(u_n) = \int_{\mathbb{R}^N} F(u_n)\,dx + \varepsilon \int_{\mathbb{R}^N} G_R(u_n)\,dx$$

and

$$\langle \Phi'_{\varepsilon,R}(u_n), v\rangle = \int_{\mathbb{R}^N} f(u_n)v\,dx + \varepsilon \int_{\mathbb{R}^N} g_R(u_n)v\,dx,$$

where $G_R(t) = G(\eta_R(t)t)$ and $g_R(t) = g(\eta_R(t)t)(\eta_R(t) + \eta'_R(t)t)$. Using (1.6), we can choose $\varepsilon_3(R) \in (0, \varepsilon_2(R))$ such that

$$4|\varepsilon| \sup_{t\in\mathbb{R}\setminus\{0\}} \left|\frac{g_R(t)}{t}\right| < \ell, \quad \text{for } |\varepsilon| \le \varepsilon_3(R), \tag{3.18}$$

where ℓ is the number in (f_2). Fix such an ε. Then by (f_2), we have, for $|t|$ small,

$$(f(t) + \varepsilon g_R(t))t \le \left(-\frac{\ell}{2} + |\varepsilon| \sup_{t\in\mathbb{R}\setminus\{0\}} \left|\frac{g_R(t)}{t}\right|\right) t^2 \le -\frac{\ell}{4}t^2 \le 0 \tag{3.19}$$

and

$$F(t) + \varepsilon G_R(t) \le \left(-\frac{\ell}{4} + \frac{|\varepsilon|}{2} \sup_{t\in\mathbb{R}\setminus\{0\}} \left|\frac{g_R(t)}{t}\right|\right) t^2 \le 0. \tag{3.20}$$

By (f_1) and (f_3), there also holds

$$\lim_{t\to\infty} \frac{(f(t) + \varepsilon g_R(t))t}{|t|^{2^*}} = \lim_{t\to\infty} \frac{F(t) + \varepsilon G_R(t)}{|t|^{2^*}} = 0. \tag{3.21}$$

By (3.19)–(3.21), we see from Lemma 3.6 that

$$\Phi_{\varepsilon,R}(u) \ge \overline{\lim_{n\to\infty}}\, \Phi_{\varepsilon,R}(u_n) = c \tag{3.22}$$

and

$$\begin{aligned}\langle \Phi'_{\varepsilon,R}(u), u\rangle &= \int_{\mathbb{R}^N} (f(u) + \varepsilon g_R(u))u\,dx \\ &\ge \overline{\lim_{n\to\infty}} \int_{\mathbb{R}^N} (f(u_n) + \varepsilon g_R(u_n))u_n\,dx = \overline{\lim_{n\to\infty}}\, \langle \Phi'_{\varepsilon,R}(u_n), u_n\rangle.\end{aligned} \tag{3.23}$$

In the rest of the proof, we use ideas from [7, Pages 362–364]. Since

$$\|(\Phi_{\varepsilon,R}|_{\mathcal{M}})'(u_n)\| \to 0,$$

we have

$$\int_{\mathbb{R}^N} \big(f(u_n) + \varepsilon g_R(u_n)\big)v\,dx = o(1)\|v\| \quad \text{if } \int_{\mathbb{R}^N} \nabla u_n \nabla v\,dx = 0.$$

This implies, for $v \in H_r^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \big(f(u_n) + \varepsilon g_R(u_n)\big) v\, dx - \theta_n \int_{\mathbb{R}^N} \nabla u_n \nabla v\, dx = o(1)\|v\|,$$

where $\theta_n = \int_{\mathbb{R}^N} \big(f(u_n) + \varepsilon g_R(u_n)\big) u_n\, dx$. Since by (3.18)

$$|f(t)+\varepsilon g_R(t)| \le C(|t|+|t|^{2^*-1}), \text{ where } C \text{ is independent of } |\varepsilon| \le \varepsilon_3(R) \text{ and } R, \quad (3.24)$$

and since $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^N)$, $\{\theta_n\}$ is bounded. We may assume that $\theta_n \to \theta$ as $n \to \infty$. Then from the last equation we see that u is a solution of

$$\theta \Delta u + f(u) + \varepsilon g_R(u) = 0.$$

By the Pohožaev identity and (3.22),

$$\theta \int_{\mathbb{R}^N} |\nabla u|^2\, dx = 2^* \int_{\mathbb{R}^N} (F(u) + \varepsilon G_R(u))\, dx \ge 2^* c > 0.$$

This implies $\theta > 0$ and $u \ne 0$. We also have

$$\theta \int_{\mathbb{R}^N} |\nabla u|^2\, dx = \int_{\mathbb{R}^N} (f(u) + \varepsilon g_R(u)) u\, dx.$$

From this and (3.23), we see that

$$\theta \int_{\mathbb{R}^N} |\nabla u|^2\, dx \ge \varlimsup_{n\to\infty} \theta_n = \theta.$$

This implies $\int_{\mathbb{R}^N} |\nabla u|^2\, dx \ge 1$. Since on the other hand $\int_{\mathbb{R}^N} |\nabla u|^2\, dx \le 1$, we obtain $\int_{\mathbb{R}^N} |\nabla u|^2\, dx = 1$, and therefore

$$\theta = \int_{\mathbb{R}^N} (f(u) + \varepsilon g_R(u)) u\, dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} (f(u_n) + \varepsilon g_R(u_n)) u_n\, dx.$$

Then $u \in \mathcal{M}$ and $\nabla u_n \to \nabla u$ strongly in $L^2(\mathbb{R}^N)$. Using (3.19) and (3.21) and applying Lemma 3.7 to $-(f(t)+\varepsilon g_R(t))t$, we see that $u_n \to u$ strongly in $L^2(\mathbb{R}^N)$, whence $u_n \to u$ strongly in $H_r^1(\mathbb{R}^N)$. The proof is complete. □

As a direct consequence of Lemmas 2.5′, 3.4 and 3.8, we have

Proposition 3.9. *Let $R > R_0$ and $|\varepsilon| \le \varepsilon_3(R)$. Then the functional $\Phi_{\varepsilon,R}|_{\mathcal{M}}$ admits m distinct critical values $\{c_{\varepsilon,R,j}\}_{j=1}^m$ satisfying (3.13) and (3.14).*

3.3 Proof of Theorem 1.1

Now we are ready to prove the first main result.

Proof of Theorem 1.1. Let $m \in \mathbb{N}$ be fixed. In the following, we always assume that $R > R_0$ and $|\varepsilon| \le \varepsilon_3(R)$, where R_0 and $\varepsilon_3(R)$ are the same as in Lemma 3.8. According to Proposition 3.9, there exists $\{u_{\varepsilon,R,j}\}_{j=1}^m \subset \mathcal{M}$ such that $\Phi_{\varepsilon,R}(u_{\varepsilon,R,j}) = c_{\varepsilon,R,j} \ge c_0 > 0$ and $(\Phi_{\varepsilon,R}|_{\mathcal{M}})'(u_{\varepsilon,R,j}) = 0$. As in the proof of Lemma 3.8, one has $\int_{\mathbb{R}^N} u_{\varepsilon,R,j}^2\,dx \le R_0 - 1$ and $\eta_R\big(|u_{\varepsilon,R,j}|_2^2\big) = 1$. Then $u_{\varepsilon,R,j}$ is a solution of

$$-\mu\Delta u = f(u) + \varepsilon g_R(u) \ \text{ in } \mathbb{R}^N$$

with $\mu = \mu_{\varepsilon,R,j}$ being a Lagrange multiplier. Using the Pohožaev identity yields

$$\mu_{\varepsilon,R,j} = 2^*\Phi_{\varepsilon,R}(u_{\varepsilon,R,j}) = 2^* c_{\varepsilon,R,j} \ge 2^* c_0.$$

Then $v_{\varepsilon,R,j}(x) = u_{\varepsilon,R,j}\big(\sqrt{\mu_{\varepsilon,R,j}}\,x\big)$ is a solution of the equation

$$-\Delta v = f(v) + \varepsilon g_R(v) \ \text{ in } \mathbb{R}^N.$$

By the Sobolev inequality,

$$\int_{\mathbb{R}^N} |v_{\varepsilon,R,j}|^{2^*}\,dx \le C\left(\int_{\mathbb{R}^N} |\nabla v_{\varepsilon,R,j}|^2\,dx\right)^{2^*/2} \le C\mu_{\varepsilon,R,j}^{-\frac{N}{2}} \le C\,(2^*c_0)^{-\frac{N}{2}}.$$

From this estimate and (3.24), using a technique due to Brezis and Kato [8] (see also [30, Lemma B.3]) we can find for any $2 < p < \infty$ a constant $C = C(p) > 0$ independent of ε, R, j, and $y \in \mathbb{R}^N$ such that $\|v_{\varepsilon,R,j}\|_{L^p(B_1(y))} \le C$. The L^p theory together with the Sobolev inequality then yields a constant $C_m > 0$ such that, for all $R > R_0$, $|\varepsilon| \le \varepsilon_3(R)$, and $1 \le j \le m$,

$$|v_{\varepsilon,R,j}|_\infty \le C_m.$$

Set $R_m = \max\{R_0, C_m\} + 1$ and denote $\varepsilon_m = \varepsilon_3(R_m)$. If $|\varepsilon| \le \varepsilon_m$, then $\{v_{\varepsilon,R_m,j}\}_{j=1}^m$ are indeed solutions of the equation (1.2). It is easy to see that (see the proof of [7, (9.18)])

$$I_\varepsilon(v_{\varepsilon,R_m,j}) = \frac{1}{N}(2^* c_{\varepsilon,R_m,j})^{-\frac{N-2}{2}},$$

where, for $v \in H_r^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$,

$$I_\varepsilon(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2\,dx - \int_{\mathbb{R}^N} F(v)\,dx - \varepsilon\int_{\mathbb{R}^N} G(v)\,dx.$$

Then, by (3.14),

$$I_\varepsilon(v_{\varepsilon,R_m,j+1}) - I_\varepsilon(v_{\varepsilon,R_m,j}) \ge 1, \quad \forall\, 1 \le j \le m-1.$$

This shows that $\{v_{\varepsilon,R_m,j}\}_{j=1}^m$ are mutually distinct. The proof is complete. □

4 Proof of Theorem 1.3

We prove Theorem 1.3 in this section. In form, equation (1.3) is the Euler equation of the functional

$$J_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right) dx - \int_{\mathbb{R}^N} F(x,u)\, dx - \varepsilon \int_{\mathbb{R}^N} G(x,u,\nabla u)\, dx.$$

But this functional is not well defined in any Sobolev space since there is no control on G for $|u|$ or $|\nabla u|$ large. We shall elaborately truncate G so that appropriate variational methods can be successfully applied to the modified functionals.

To obtain solutions of (1.3) by considering J_ε, we need first to study behaviors of $J := J_0$. Throughout this section we shall work in $E := H^1_r(\mathbb{R}^N)$ with the norm

$$\|u\|_V = \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\, dx\right)^{\frac{1}{2}}.$$

Under the assumptions of Theorem 1.3 it is easy to see that $J \in C^1(E,\mathbb{R})$ is an even functional.

4.1 Essential values of J

In this subsection, we show that J has an unbounded sequence of positive essential values. We first prove the following abstract result which will be used both in this section and in Section 5.

Theorem 4.1. *Let E be an infinite dimensional Banach space and $I \in C^1(E,\mathbb{R})$ be even, satisfy the (PS) condition, and $I(0) = 0$. If $E = V \oplus X$, where V is finite dimensional, and I satisfies*

(a) *there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho \cap X} \geq \alpha$, and*

(b) *there exist an increasing sequence of subspaces $E_n \subset E$ with $\dim E_n = n$ and an increasing sequence of positive numbers R_n such that $I \leq 0$ on $E_n \backslash B_{R_n}$,*

then I has an unbounded sequence of positive essential values.

Under the assumptions of Theorem 4.1, [27, Theorem 9.12] asserts that I possesses an unbounded sequence of positive critical values. In the statement of [27, Theorem 9.12], an assumption (I_2') stronger than (b) is assumed. But upon a careful inspection of the proof of [27, Theorem 9.12], one easily sees that (b) is a valid replacement of (I_2') there. Theorem 4.1 strengthens the conclusion of [27, Theorem 9.12] since essential values are nondegenerate in the sense that they are inherited under small perturbation of the functional while critical values not. The proof of Theorem 4.1 is based on that of [27, Theorem 9.12].

Proof of Theorem 4.1. Since $\{E_n\}$ is an increasing sequence of subspaces of E with $\dim E_n = n$, there exists a linearly independent sequence $\{e_n\}_1^\infty \subset E$ such that $E_n = \mathrm{span}\{e_1, e_2, \cdots, e_n\}$. We use Σ to denote the set of compact and symmetric subsets of E.

Let us recall the critical values constructed in [27]. Set $D_n := B_{R_n} \cap E_n$. Let

$$G_n := \{h \in C(D_n, E) \,|\, h \text{ is odd and } h = id \text{ on } \partial B_{R_n} \cap E_n\}.$$

Then $G_n \neq \emptyset$. Set

$$\Gamma_j := \{h(\overline{D_n \backslash Y}) \,|\, h \in G_n,\ n \geq j,\ Y \in \Sigma,\ \text{and } \gamma(Y) \leq n - j\}.$$

Define

$$c_j = \inf_{B \in \Gamma_j} \max_{u \in B} I(u), \quad j \in \mathbb{N}.$$

According to [27, Corollary 9.29, Propositions 9.30 and 9.33], for $j > k := \dim V$, $c_j \geq \alpha$, c_j are critical values of I, and $\lim_{j\to\infty} c_j = +\infty$.

Define a new sequence of minimax values

$$b_j = \inf_{h \in G_j} \max_{u \in D_j} I(h(u)), \quad j \in \mathbb{N}.$$

Since $h(D_j) \in \Gamma_j$ for $h \in G_j$, we have $b_j \geq c_j$. Then $\lim_{j\to\infty} b_j = +\infty$. Set

$$\Lambda = \{c \in \mathbb{R} \,|\, c \text{ is an essential value of } I\}.$$

We need to prove that $\Lambda \neq \emptyset$ and $\sup \Lambda = +\infty$.

Assume to the contrary that either $\Lambda = \emptyset$ or $\sup \Lambda < +\infty$. Then since $\lim_{j\to\infty} b_j = +\infty$ there exists $j > k$ such that $\alpha \leq b_j < b_{j+1}$ and $\Lambda \cap [b_j, +\infty) = \emptyset$. We fix three numbers β', β, $\beta'' \in \mathbb{R}$ such that $0 < b_j < \beta' < \beta < \beta'' < b_{j+1}$ and choose $h \in G_j$ such that

$$\max_{u \in D_j} I(h(u)) < \beta'.$$

Set

$$D_{j+1}^+ = \{u \in E_{j+1} \,|\, u = v + te_{j+1} \text{ with } v \in E_j \text{ and } t \geq 0, \|u\|_V \leq R_{j+1}\}$$

and denote by ∂D_{j+1}^+ the boundary of D_{j+1}^+ in E_{j+1}. Extend h to a continuous map $h_1 : \partial D_{j+1}^+ \to E$ defined by

$$h_1(u) = \begin{cases} h(u), & \text{if } u \in D_j, \\ u, & \text{if } u \in \partial D_{j+1}^+ \setminus D_j. \end{cases}$$

Clearly, $h_1(\partial D_{j+1}^+) \subset I^{\beta'}$. Extend h_1 to a map $h_2 \in C(D_{j+1}^+, E)$ and choose $\nu \in \mathbb{R}$ such that

$$\nu > \max\left\{b_{j+1}, \max_{u \in D_{j+1}^+} I(h_2(u))\right\}.$$

By Lemma 2.3, the pair $(I^{+\infty}, I^{\beta})$ is trivial. So there exist two closed subsets A, B of E such that $I^{\beta'} \subseteq A \subseteq I^{\beta''}$ and $I^{\nu} \subseteq B$, and there is a strong deformation retraction $\eta : [0,1] \times B \to B$ from B to A. Define $h_3 \in C(D_{j+1}^{+}, E)$ by $h_3(u) = \eta(1, h_2(u))$, which has the following properties:

$$h_3(D_{j+1}^{+}) \subset I^{\beta''}, \quad h_3 \text{ is odd on } D_{j+1}^{+} \cap E_j, \quad h_3 = id \text{ on } \partial D_{j+1}^{+} \cap \partial D_{j+1}. \tag{4.1}$$

Define $h_4 \in C(D_{j+1}, E)$ by

$$h_4(u) = \begin{cases} h_3(u), & \text{if } u \in D_{j+1}^{+}, \\ -h_3(-u), & \text{if } u \in D_{j+1} \setminus D_{j+1}^{+}. \end{cases}$$

Clearly, h_4 is odd and $h_4|_{\partial D_{j+1}} = id$, which means that $h_4 \in G_{j+1}$ and thus

$$b_{j+1} \leq \max_{u \in D_{j+1}} I(h_4(u)).$$

But we see from (4.1) that

$$\max_{u \in D_{j+1}} I(h_4(u)) = \max_{u \in D_{j+1}^{+}} I(h_3(u)) \leq \beta'' < b_{j+1},$$

yielding a contradiction. The proof is finished. □

Using Theorem 4.1, we prove the following result for the functional J defined at the beginning of this section.

Lemma 4.2. *J has an unbounded sequence of positive essential values.*

Proof. We need to verify that J satisfies all the conditions in Theorem 4.1. From the assumptions (V) and (F_1)–(F_4), it is standard to show that $J \in C^1(E, \mathbb{R})$ is even, J satisfies the (PS) condition, $J(0) = 0$, and J satisfies (a) with $X = E$.

By (F_5), there exist $r, \delta > 0$ such that $F(x, t_0) \geq \delta$ for $|x - x_0| \leq r$. Then using (F_1)–(F_4) again, we can find a constant $C_1 > 0$ such that

$$F(x,t) \geq \delta |t/t_0|^{\mu} - C_1 |t|^2, \quad \forall\, |x - x_0| \leq r,\ t \in \mathbb{R}.$$

This implies

$$J(u) \leq C_2 \|u\|_V^2 - \frac{\delta}{|t_0|^{\mu}} |u|_{\mu}^{\mu}$$

for all $u \in C_0^{\infty}(B_{|x_0|+r} \backslash B_{|x_0|-r})$ and for some constant $C_2 > 0$. Let $\{e_n\}_{n=1}^{\infty} \subset C_0^{\infty}(B_{|x_0|+r} \backslash B_{|x_0|-r})$ be a linearly independent sequence of radial functions and set $E_n = \text{span}\{e_1, e_2, \cdots, e_n\}$. Then it is easy to see that J satisfies (b). □

4.2 Critical values of $J_{\varepsilon,R,\theta}$

Let $R > 1$ be arbitrary but fixed. In what follows, C_R denotes positive constants depending only on R which may be variant from line to line. Assumption (G) implies

$$|G(x,t,\xi)| \le C_R t^2, \quad \forall\, x \in \mathbb{R}^N,\ |t| \le R,\ |\xi| \le R. \tag{4.2}$$

Recall the cut off function $\eta_R \in C^\infty(\mathbb{R},[0,1])$ defined in Section 3 which satisfies

$$\eta_R(t) = 1 \text{ if } |t| \le R-1, \quad \eta_R(t) = 0 \text{ if } |t| \ge R, \quad |\eta_R'(t)| \le 2. \tag{4.3}$$

Let $0 < \theta < 2/(N-2)$ be arbitrary but fixed. Define a modified functional $J_{\varepsilon,R,\theta} : E \to \mathbb{R}$ as

$$J_{\varepsilon,R,\theta}(u) = J(u) - \varepsilon \eta_R\big(|u|_{2+2\theta}^{2+2\theta}\big) \int_{\mathbb{R}^N} G^R(x,u,\nabla u)\, dx,$$

where

$$G^R(x,t,\xi) = G\Big(x,\ \eta_R\big(|t|^{1+\theta}\big)|t|^\theta t,\ \eta_R\big(|\xi|^2\big)\xi\Big).$$

By (4.2) and (4.3), we have

$$|G^R(x,t,\xi)| \le C_R |t|^{2+2\theta}, \quad \forall\, x \in \mathbb{R}^N,\ t \in \mathbb{R},\ \xi \in \mathbb{R}^N. \tag{4.4}$$

This implies

$$\left| \int_{\mathbb{R}^N} G^R(x,u,\nabla u)\, dx \right| \le C_R |u|_{2+2\theta}^{2+2\theta}, \quad \text{for } u \in E. \tag{4.5}$$

Using (4.5), we obtain the following estimate on the difference between J and $J_{\varepsilon,R,\theta}$.

Lemma 4.3. *There exists $C_R > 0$ independent of ε and θ such that*

$$\sup_{u \in E} |J_{\varepsilon,R,\theta}(u) - J(u)| \le |\varepsilon| C_R.$$

For simplicity of notation, we denote

$$\begin{aligned} a(x,t,\xi) = {} & \eta_R\big(|\xi|^2\big) G_\xi\Big(x, \eta_R\big(|t|^{1+\theta}\big)|t|^\theta t, \eta_R\big(|\xi|^2\big)\xi\Big) \\ & + 2\eta_R'\big(|\xi|^2\big) \Big[G_\xi\Big(x, \eta_R\big(|t|^{1+\theta}\big)|t|^\theta t, \eta_R\big(|\xi|^2\big)\xi\Big) \cdot \xi \Big] \xi, \end{aligned}$$

and

$$b(x,t,\xi) = G_t\Big(x, \eta_R\big(|t|^{1+\theta}\big)|t|^\theta t, \eta_R\big(|\xi|^2\big)\xi\Big) \left(\eta_R'\big(|t|^{1+\theta}\big)|t|^{1+2\theta} + \eta_R\big(|t|^{1+\theta}\big)|t|^\theta\right).$$

Set

$$a(x,t,\xi) = (a_1(x,t,\xi), a_2(x,t,\xi), \cdots, a_N(x,t,\xi)).$$

We give some basic estimates on a and b which will be used in the subsequent argument.

Lemma 4.4. *There exists $C_R > 0$ independent of θ such that, for $x, \xi \in \mathbb{R}^N$ and $t \in \mathbb{R}$,*

(a) $|a(x,t,\xi)| \le C_R(\min\{|t|, |t|^{1+\theta}\} + |\xi|)$,

(b) $|b(x,t,\xi)| \le C_R \min\{|t|, |t|^{1+2\theta}\}$,

(c) $\left|\frac{\partial a_i(x,t,\xi)}{\partial x_j}\right| + \left|\frac{\partial a_i(x,t,\xi)}{\partial t}\right| + \left|\frac{\partial a_i(x,t,\xi)}{\partial \xi_j}\right| \le C_R, \quad \forall\, 1 \le i,\, j \le N.$

Proof. This follows easily from (4.3) and assumption (G). We omit the details. □

Using Lemma 4.4(a)–(b) together with (4.4), it is easy to show that $J_{\varepsilon,R,\theta} \in C^1(E, \mathbb{R})$ and

$$\begin{aligned}\langle J'_{\varepsilon,R,\theta}(u), \varphi\rangle = &\int_{\mathbb{R}^N} (\nabla u \nabla \varphi + V(x) u\varphi)\, dx - \int_{\mathbb{R}^N} f(x,u)\varphi\, dx \\ &- \varepsilon(2+2\theta)\eta_R'\big(|u|_{2+2\theta}^{2+2\theta}\big) \int_{\mathbb{R}^N} |u|^{2\theta} u\varphi\, dx \int_{\mathbb{R}^N} G^R(x,u,\nabla u)\, dx \\ &- \varepsilon \eta_R\big(|u|_{2+2\theta}^{2+2\theta}\big) \int_{\mathbb{R}^N} a(x,u,\nabla u)\nabla\varphi\, dx \\ &- \varepsilon(1+\theta)\eta_R\big(|u|_{2+2\theta}^{2+2\theta}\big) \int_{\mathbb{R}^N} b(x,u,\nabla u)\varphi\, dx;\end{aligned}$$

one may consult [30, Appendix C] for a similar argument.

The next result asserts that the (PS) condition holds for $J_{\varepsilon,R,\theta}$. The reason why we introduce the parameter θ in the definition of $J_{\varepsilon,R,\theta}$ is to guarantee the validity of it.

Lemma 4.5. *There exists a number $\varepsilon_1(R) > 0$ independent of θ such that, if $|\varepsilon| \le \varepsilon_1(R)$, then the functional $J_{\varepsilon,R,\theta}$ satisfies the $(PS)_c$ condition for every $c \in \mathbb{R}$.*

Proof. Let $\{u_n\} \subset E$ be a $(PS)_c$ sequence, that is, $J_{\varepsilon,R,\theta}(u_n) \to c$ and $\|J'_{\varepsilon,R,\theta}(u_n)\| \to 0$ as $n \to \infty$. Then, by (F_4), we have

$$\begin{aligned}J_{\varepsilon,R,\theta}(u_n) - &\frac{1}{\mu}\langle J'_{\varepsilon,R,\theta}(u_n), u_n\rangle \\ &\ge \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|_V^2 - \varepsilon\eta_R\big(|u_n|_{2+2\theta}^{2+2\theta}\big)\int_{\mathbb{R}^N} G^R(x,u_n,\nabla u_n)\, dx \\ &\quad + \frac{\varepsilon(2+2\theta)}{\mu}\eta_R'\big(|u_n|_{2+2\theta}^{2+2\theta}\big)|u_n|_{2+2\theta}^{2+2\theta}\int_{\mathbb{R}^N} G^R(x,u_n,\nabla u_n)\, dx \\ &\quad + \frac{\varepsilon}{\mu}\eta_R\big(|u_n|_{2+2\theta}^{2+2\theta}\big)\int_{\mathbb{R}^N} a(x,u_n,\nabla u_n)\nabla u_n\, dx \\ &\quad + \frac{\varepsilon(1+\theta)}{\mu}\eta_R\big(|u_n|_{2+2\theta}^{2+2\theta}\big)\int_{\mathbb{R}^N} b(x,u_n,\nabla u_n)u_n\, dx.\end{aligned}$$

It follows from Lemma 4.4(a)–(b) that

$$\left|\int_{\mathbb{R}^N} a(x,u_n,\nabla u_n)\nabla u_n\, dx\right| + \left|\int_{\mathbb{R}^N} b(x,u_n,\nabla u_n)u_n\, dx\right| \le C_R\|u_n\|_V^2,$$

which combined with (4.3) and (4.5) leads to

$$\begin{aligned} C + \|u_n\|_V &\ge J_{\varepsilon,R,\theta}(u_n) - \frac{1}{\mu}\langle J'_{\varepsilon,R,\theta}(u_n), u_n\rangle \\ &\ge \frac{\mu-2}{2\mu}\|u_n\|_V^2 - C'_R|\varepsilon| - C''_R|\varepsilon|\|u_n\|_V^2 \end{aligned}$$

for n large. Here $C'_R, C''_R > 0$ depend only on R. If $|\varepsilon| \le \varepsilon'_1(R) := \min\{1/C'_R, (\mu-2)/(4\mu C''_R)\}$, then $C + \|u_n\|_V \ge \frac{\mu-2}{4\mu}\|u_n\|_V^2 - 1$, which shows that $\{u_n\}$ is bounded in E. By extracting a subsequence, one can assume that $u_n \to u$ weakly in E, strongly in $L^q(\mathbb{R}^N)$ for $2 < q < 2^*$, and a.e. in $\mathbb{R}^N$.

Next we show that $u_n \to u$ strongly in E. We have

$$\begin{aligned} &\langle J'_{\varepsilon,R,\theta}(u_n) - J'_{\varepsilon,R,\theta}(u), u_n - u\rangle \\ &\quad = \|u_n - u\|_V^2 - \int_{\mathbb{R}^N} \big(f(x,u_n) - f(x,u)\big)(u_n - u)\,dx \\ &\qquad - \varepsilon(2+2\theta)\eta'_R\big(|u_n|_{2+2\theta}^{2+2\theta}\big)\int_{\mathbb{R}^N} |u_n|^{2\theta}u_n(u_n-u)\,dx \int_{\mathbb{R}^N} G^R(x,u_n,\nabla u_n)\,dx \qquad (T_1) \\ &\qquad + \varepsilon(2+2\theta)\eta'_R\big(|u|_{2+2\theta}^{2+2\theta}\big)\int_{\mathbb{R}^N} |u|^{2\theta}u(u_n-u)\,dx \int_{\mathbb{R}^N} G^R(x,u,\nabla u)\,dx \qquad (T_2) \\ &\qquad - \varepsilon\eta_R\big(|u_n|_{2+2\theta}^{2+2\theta}\big)\int_{\mathbb{R}^N} a(x,u_n,\nabla u_n)\nabla(u_n-u)\,dx \qquad (T_3) \\ &\qquad + \varepsilon\eta_R\big(|u|_{2+2\theta}^{2+2\theta}\big)\int_{\mathbb{R}^N} a(x,u,\nabla u)\nabla(u_n-u)\,dx \qquad (T_4) \\ &\qquad - \varepsilon(1+\theta)\eta_R\big(|u_n|_{2+2\theta}^{2+2\theta}\big)\int_{\mathbb{R}^N} b(x,u_n,\nabla u_n)(u_n-u)\,dx \qquad (T_5) \\ &\qquad + \varepsilon(1+\theta)\eta_R\big(|u|_{2+2\theta}^{2+2\theta}\big)\int_{\mathbb{R}^N} b(x,u,\nabla u)(u_n-u)\,dx. \qquad (T_6) \end{aligned}$$

Since $u_n \to u$ strongly in $L^q(\mathbb{R}^N)$ for $2 < q < 2^*$, from (F_2) and (F_3), a standard argument shows that

$$\int_{\mathbb{R}^N} \big(f(x,u_n) - f(x,u)\big)(u_n-u)\,dx = o(1).$$

By (4.3) and (4.5), we have $T_1 = o(1)$ and $T_2 = o(1)$ as $n \to \infty$. In order to estimate $T_3 + T_4$, we observe that

$$\begin{aligned} |T_3 + T_4| \le{}& |\varepsilon|\,\big|\eta_R\big(|u_n|_{2+2\theta}^{2+2\theta}\big) - \eta_R\big(|u|_{2+2\theta}^{2+2\theta}\big)\big|\left|\int_{\mathbb{R}^N} a(x,u_n,\nabla u_n)\nabla(u_n-u)\,dx\right| \\ &+ |\varepsilon|\left|\int_{\mathbb{R}^N} (a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u))\nabla(u_n-u)\,dx\right| \\ &+ |\varepsilon|\left|\int_{\mathbb{R}^N} (a(x,u_n,\nabla u) - a(x,u,\nabla u))\nabla(u_n-u)\,dx\right|. \qquad (4.6) \end{aligned}$$

Clearly, $\eta_R\big(|u_n|_{2+2\theta}^{2+2\theta}\big) - \eta_R\big(|u|_{2+2\theta}^{2+2\theta}\big) = o(1)$ as $n \to \infty$. By Lemma 4.4(a), there holds

$$\left| \int_{\mathbb{R}^N} a(x, u_n, \nabla u_n) \nabla (u_n - u)\, dx \right| \le C_R \|u_n\|_V \|u_n - u\|_V \le C_R.$$

Using Lemma 4.4(c), we see that

$$\left| \int_{\mathbb{R}^N} (a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)) \nabla (u_n - u)\, dx \right| \le C_R \|u_n - u\|_V^2.$$

Since, by Lemma 4.4(a), $a(x, u_n, \nabla u) \to a(x, u, \nabla u)$ in $L^2(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} (a(x, u_n, \nabla u) - a(x, u, \nabla u)) \nabla (u_n - u)\, dx = o(1), \quad \text{as } n \to \infty.$$

Putting the above estimates into (4.6) yields

$$|T_3 + T_4| \le |\varepsilon| C_R \|u_n - u\|_V^2 + o(1), \quad \text{as } n \to \infty.$$

Finally, we estimate T_5 and T_6. By Lemma 4.4(b) and the Hölder inequality,

$$\left| \int_{\mathbb{R}^N} b(x, u_n, \nabla u_n)(u_n - u)\, dx \right| \le C_R |u_n|_{2+2\theta}^{1+2\theta} |u_n - u|_{2+2\theta} = o(1),$$

which implies that $T_5 = o(1)$ as $n \to \infty$. Similarly, $T_6 = o(1)$ as $n \to \infty$. Therefore,

$$\begin{aligned} o(1) &= \langle J'_{\varepsilon,R,\theta}(u_n) - J'_{\varepsilon,R,\theta}(u), u_n - u \rangle \\ &\ge \|u_n - u\|_V^2 - |\varepsilon| C_R \|u_n - u\|_V^2 + o(1). \end{aligned}$$

If $|\varepsilon| \le \varepsilon_1(R) := \min\{\varepsilon_1'(R), 1/(2C_R)\}$, then $u_n \to u$ strongly in E. □

Let $m \in \mathbb{N}$ be arbitrary but fixed. The main result in this subsection is

Proposition 4.6. *There exists a number $\varepsilon_2(R) \in (0, \varepsilon_1(R))$ independent of θ such that if $|\varepsilon| \le \varepsilon_2(R)$ then the functional $J_{\varepsilon,R,\theta}$ has m critical values $\{c_{\varepsilon,R,\theta,j}\}_{j=1}^m$ satisfying*

$$0 < c_0^{-1} \le c_{\varepsilon,R,\theta,j} \le c_0, \quad \forall\, 1 \le j \le m$$

and

$$c_{\varepsilon,R,\theta,j+1} - c_{\varepsilon,R,\theta,j} \ge 1, \quad \forall\, 1 \le j \le m-1,$$

where $c_0 = c_0(m)$ is independent of ε, R and θ.

Proof. By Lemma 4.2, there exist m essential values $\{d_j\}_{j=1}^m$ of J such that

$$d_1 > 1, \quad d_{j+1} - d_j > 3 \text{ for } 1 \le j \le m-1.$$

According to Lemmas 2.4 and 4.3, there is a number $\varepsilon_2(R) \in (0, \varepsilon_1(R))$ independent of θ such that, if $|\varepsilon| \le \varepsilon_2(R)$, then $J_{\varepsilon,R,\theta}$ has m essential values $\{c_{\varepsilon,R,\theta,j}\}_{j=1}^m$ satisfying

$$|c_{\varepsilon,R,\theta,j} - d_j| < 1, \quad \forall\, 1 \le j \le m.$$

Then

$$0 < d_1 - 1 \le c_{\varepsilon,R,\theta,j} \le d_m + 1, \quad \forall\, 1 \le j \le m$$

and

$$c_{\varepsilon,R,\theta,j+1} - c_{\varepsilon,R,\theta,j} \ge 1, \quad \forall\, 1 \le j \le m-1.$$

We see from Lemmas 2.5 and 4.5 that $\{c_{\varepsilon,R,\theta,j}\}_{j=1}^m$ are m distinct critical values of $J_{\varepsilon,R,\theta}$. The proof is finished. □

Let $m \in \mathbb{N}$ be fixed and $|\varepsilon| \le \varepsilon_2(R)$. By Proposition 4.6, there exist m elements $u_{\varepsilon,R,\theta,j} \in E$ such that

$$J_{\varepsilon,R,\theta}\big(u_{\varepsilon,R,\theta,j}\big) = c_{\varepsilon,R,\theta,j} \quad \text{and} \quad J'_{\varepsilon,R,\theta}\big(u_{\varepsilon,R,\theta,j}\big) = 0, \quad j = 1, 2, \cdots, m.$$

4.3 L^∞ estimate for $u_{\varepsilon,R,\theta,j}$

Note that $u_{\varepsilon,R,\theta,j}$ are weak solutions of the equation

$$\begin{aligned} -\Delta u + V(x)u = f(x,u) + \Big[\varepsilon(2+2\theta)\eta_R'\big(|u|_{2+2\theta}^{2+2\theta}\big)\int_{\mathbb{R}^N} G^R(x,u,\nabla u)\,dx\Big]|u|^{2\theta}u \\ - \varepsilon\eta_R\big(|u|_{2+2\theta}^{2+2\theta}\big)\operatorname{div}[a(x,u,\nabla u)] + \varepsilon(1+\theta)\eta_R\big(|u|_{2+2\theta}^{2+2\theta}\big)b(x,u,\nabla u). \end{aligned}$$

We first provide uniform estimates on the Sobolev norm $\|u_{\varepsilon,R,\theta,j}\|_V$.

Lemma 4.7. *There exists $C > 0$ independent of ε, R and θ such that*

$$\|u_{\varepsilon,R,\theta,j}\|_V \le C, \quad \forall\, 1 \le j \le m.$$

Proof. Write $u_j := u_{\varepsilon,R,\theta,j}$ for simplicity. Observing the proof of Lemma 4.5, we see that

$$c_{\varepsilon,R,\theta,j} = J_{\varepsilon,R,\theta}(u_j) - \frac{1}{\mu}\langle J'_{\varepsilon,R,\theta}(u_j), u_j\rangle \ge \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_j\|_V^2 - C_R'|\varepsilon| - C_R''|\varepsilon|\|u_j\|_V^2.$$

Since $|\varepsilon| \le \varepsilon_2(R) < \varepsilon_1(R)$ implies $C_R'|\varepsilon| < 1$ and $C_R''|\varepsilon| < \frac{\mu-2}{4\mu}$, by Proposition 4.6, we have the desired estimate. □

By Lemma 4.7 and the Sobolev inequality, we have

$$|u_{\varepsilon,R,\theta,j}|_{2+2\theta}^{2+2\theta} \le C_1, \quad \forall\, 1 \le j \le m,$$

where $C_1 > 0$ is independent of ε, R and θ. Denote $R_0 := C_1 + 1$ and let $R > R_0$. Then, by the definition of η_R, $u_{\varepsilon,R,\theta,j}$ $(1 \le j \le m)$ are weak solutions of the equation

$$-\Delta u + V(x)u = f(x,u) - \varepsilon \operatorname{div}[a(x,u,\nabla u)] + \varepsilon(1+\theta)b(x,u,\nabla u). \tag{4.7}$$

To estimate $|u_{\varepsilon,R,\theta,j}|_\infty$, we shall use the following lemma which is a special case of [18, Lemma 5.4 in Chapter 2].

Lemma 4.8. *Let $u \in H^1(\mathbb{R}^N)$ and $y \in \mathbb{R}^N$. For $k > 1$ and $\rho \in [1, 2]$ we denote*

$$A_{k,\rho} = \{x \in B_\rho(y) \,|\, u(x) > k\}.$$

Assume that for any $k > 1$ and any $1 \le \rho_1 < \rho_2 \le 2$, the function $u(x)$ satisfies the inequalities

$$\int_{A_{k,\rho_1}} |\nabla u|^2 \, dx \le \gamma \Big[(\rho_2 - \rho_1)^{-2} \int_{A_{k,\rho_2}} (u - k)^2 \, dx + \mathrm{meas}^{1 - \frac{2}{N} + \varepsilon}(A_{k,\rho_2}) \Big],$$

where $\gamma > 0$ and $\varepsilon \in (0, 2/N]$ are positive constants. Then $\sup_{x \in B_1(y)} u(x) \le C$ where C is a constant depending only on N, γ, ε, and $\|u\|_{L^2(B_2(y))}$.

We also quote a result from [20] which will be used to estimate $|\nabla u_{\varepsilon,R,\theta,j}|_\infty$. Consider an equation of the form

$$\mathrm{div}\, [A(x, u, \nabla u)] + B(x, u, \nabla u) = 0, \tag{4.8}$$

where $A \in C^1(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N, \mathbb{R}^N)$ and $B \in C(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N, \mathbb{R})$. Let

$$A(x, t, \xi) = (A^1(x, t, \xi),\ A^2(x, t, \xi) \cdots, A^N(x, t, \xi))$$

and set

$$a^{ij}(x, t, \xi) = \frac{\partial A^i}{\partial \xi_j}(x, t, \xi).$$

The following lemma is a special case of [20, Theorem 1.7], where we in particular choose $g(t) = \frac{t}{2}$.

Lemma 4.9. *Suppose that the following conditions are satisfied*

$$a^{ij}(x, t, \xi) y_i y_j \ge \frac{1}{2} |y|^2,$$

$$|a^{ij}(x, t, \xi)| \le \Lambda,$$

$$|A(x, t, \xi) - A(x', t', \xi)| \le \Lambda(|x - x'| + |t - t'|),$$

$$|B(x, t, \xi)| \le \Lambda,$$

for some positive constant Λ whenever $x, x', \xi, y \in \mathbb{R}^N$ and $t, t' \in [-M_0, M_0]$ for some positive constant M_0. Then any $D^{1,2}(\mathbb{R}^N)$ solution u of (4.8) with $|u|_\infty \le M_0$ is in $C^{1,\beta}(\mathbb{R}^N)$ for some positive constant $\beta = \beta(\Lambda, N)$ and

$$\|u\|_{C^{1,\beta}(\mathbb{R}^N)} \le C(\Lambda, N, M_0).$$

Lemma 4.10. *Let R_0 be the number defined before (4.7). For $R > R_0$ there exists $\varepsilon_3(R) \in (0, \varepsilon_2(R))$ independent of θ such that if $|\varepsilon| \le \varepsilon_3(R)$ then*

$$|u_{\varepsilon,R,\theta,j}|_\infty + |\nabla u_{\varepsilon,R,\theta,j}|_\infty \le C, \quad \forall\, 1 \le j \le m,$$

where $C > 0$ is independent of ε, R and θ.

Proof. Write $u_j := u_{\varepsilon,R,\theta,j}$ for simplicity. We first estimate $|u_j|_\infty$ and this is divided into two steps. We only consider $N \geq 3$ since the argument is easier for $N = 2$. In the first step we show that

$$|u_j|_\nu \leq C_1$$

for some constants $C_1 > 0$ and $\nu > \frac{Np}{2}$ independent of ε, R and θ. By Lemma 4.4(a)–(b), we have

$$\begin{aligned}&\int_{\mathbb{R}^N} \big(\nabla u_j \nabla \varphi + V(x) u_j \varphi\big)\, dx \\ &\quad \leq \int_{\mathbb{R}^N} f(x, u_j)\varphi\, dx + |\varepsilon| C_R \int_{\mathbb{R}^N} \big(|\nabla u_j||\nabla \varphi| + |u_j||\nabla \varphi| + |u_j \varphi|\big)\, dx \end{aligned} \tag{4.9}$$

for $\varphi \in E$. We set $u_j^M = \max\big\{-M, \min\big\{u_j, M\big\}\big\}$ for $M > 0$ and choose $\varphi = \big|u_j^M\big|^s u_j^M$ with $s > 0$ as a test function in (4.9) and then let $M \to +\infty$. It follows from (F_2) and (F_3) that

$$\begin{aligned}&(s+1)\int_{\mathbb{R}^N} |u_j|^s |\nabla u_j|^2\, dx + \int_{\mathbb{R}^N} V(x)|u_j|^{s+2}\, dx \\ &\quad \leq \Big(\frac{\alpha_0}{4} + |\varepsilon| C_R + \frac{1}{2}|\varepsilon| C_R(s+1)\Big) \int_{\mathbb{R}^N} |u_j|^{s+2}\, dx + C_2 \int_{\mathbb{R}^N} |u_j|^{s+p}\, dx \\ &\qquad + \frac{3}{2}|\varepsilon| C_R(s+1) \int_{\mathbb{R}^N} |u_j|^s |\nabla u_j|^2\, dx,\end{aligned}$$

where $\alpha_0 = \inf_{x\in\mathbb{R}^N} V(x)$ and where we have used the inequality

$$|u_j|^{s+1}|\nabla u_j| \leq \frac{1}{2}(|u_j|^s|\nabla u_j|^2 + |u_j|^{s+2}).$$

Define

$$s_0 = 2^* - p, \quad s_n = \frac{N}{N-2}(s_{n-1} + 2) - p \quad \text{for} \quad n = 1, 2, \cdots,$$

and fix an integer n_0 such that $s_{n_0} > \frac{(N-2)p}{2}$. Let

$$\varepsilon_3'(R) = \min\left\{\varepsilon_2(R), \frac{1}{3C_R}, \frac{\alpha_0}{2C_R(s_{n_0} + 1)}\right\}.$$

Then for $|\varepsilon| \leq \varepsilon_3'(R)$ and $s_0 \leq s \leq s_{n_0}$,

$$\left(\int_{\mathbb{R}^N} |u_j|^{\frac{(s+2)N}{N-2}}\, dx\right)^{\frac{N-2}{N}} \leq C_N \int_{\mathbb{R}^N} \big|u_j\big|^s \big|\nabla u_j\big|^2\, dx \leq C_3 \int_{\mathbb{R}^N} |u_j|^{p+s}\, dx.$$

From this inequality, doing n_0 steps of iteration we have $|u_j|_\nu \leq C_1$ for some constants $C_1 > 0$ and $\nu := s_{n_0} + p > \frac{Np}{2}$ independent of ε, R and θ, as claimed above.

In the second step we estimate $|u_j|_\infty$. Let $y \in \mathbb{R}^N$. For $1 \leq \rho_1 < \rho_2 \leq 2$, choose $\eta \in C_0^\infty(B_{\rho_2}(y))$ such that $0 \leq \eta \leq 1$, $|\nabla \eta| \leq 2/(\rho_2 - \rho_1)$, and $\eta(x) = 1$ for $x \in B_{\rho_1}(y)$. For $k > 1$ and $\rho \in [1, 2]$ set

$$A_{k,\rho} = \{x \in B_\rho(y) \,|\, u_j(x) > k\}.$$

Using $(u_j-k)^+\eta^2$ as a test function in (4.7), we have

$$\begin{aligned}\int_{A_{k,\rho_2}} \nabla u_j \nabla((u_j-k)\eta^2)\,dx + \int_{A_{k,\rho_2}} V(x)u_j(u_j-k)\eta^2\,dx \\ = \int_{A_{k,\rho_2}} f(x,u_j)(u_j-k)\eta^2\,dx + \varepsilon\int_{A_{k,\rho_2}} a\nabla((u_j-k)\eta^2)\,dx \\ + \varepsilon(1+\theta)\int_{A_{k,\rho_2}} b(u_j-k)\eta^2\,dx.\end{aligned}$$

The left hand side can be estimated as

$$\begin{aligned}\text{LHS} =& \int_{A_{k,\rho_2}} \eta^2|\nabla u_j|^2\,dx + 2\int_{A_{k,\rho_2}} \eta(u_j-k)\nabla u_j\nabla\eta\,dx + \int_{A_{k,\rho_2}} V(x)u_j(u_j-k)\eta^2\,dx \\ \geq& \frac{1}{2}\int_{A_{k,\rho_2}} \eta^2|\nabla u_j|^2\,dx - 2\int_{A_{k,\rho_2}} (u_j-k)^2|\nabla\eta|^2\,dx + \alpha_0\int_{A_{k,\rho_2}} u_j(u_j-k)\eta^2\,dx.\end{aligned}$$

For the right hand side, by (F_2), (F_3) and Lemma 4.4(a)–(b), we have

$$\begin{aligned}\text{RHS} \leq& \frac{\alpha_0}{2}\int_{A_{k,\rho_2}} u_j(u_j-k)\eta^2\,dx + C_4\int_{A_{k,\rho_2}} u_j^p\eta^2\,dx \\ &+ |\varepsilon|C_R\int_{A_{k,\rho_2}} (u_j+|\nabla u_j|)|\nabla((u_j-k)\eta^2)|\,dx + |\varepsilon|C_R\int_{A_{k,\rho_2}} u_j(u_j-k)\eta^2\,dx,\end{aligned}$$

where $C_4>0$ is a constant independent of ε, R, θ, and y. Let

$$\varepsilon_3''(R) = \min\left\{\varepsilon_3'(R), \frac{1}{10C_R}, \frac{\alpha_0}{2C_R}\right\}.$$

Since on A_{k,ρ_2},

$$(u_j+|\nabla u_j|)|\nabla((u_j-k)\eta^2)| \leq \frac{5}{2}\eta^2|\nabla u_j|^2 + \frac{3}{2}\eta^2u_j^2 + 2(u_j-k)^2|\nabla\eta|^2,$$

we see that, for $|\varepsilon|\leq\varepsilon_3''(R)$,

$$\int_{A_{k,\rho_2}} \eta^2|\nabla u_j|^2\,dx \leq 12\int_{A_{k,\rho_2}} (u_j-k)^2|\nabla\eta|^2\,dx + C_5\int_{A_{k,\rho_2}} u_j^p\eta^2,$$

for some constant $C_5>0$ independent of ε, R, θ, and y. This implies

$$\begin{aligned}\int_{A_{k,\rho_1}} |\nabla u_j|^2\,dx \leq& \frac{48}{(\rho_2-\rho_1)^2}\int_{A_{k,\rho_2}} (u_j-k)^2\,dx + C_5|u_j|_\nu^p \text{meas}^{1-\frac{p}{\nu}}(A_{k,\rho_2}) \\ \leq& \frac{48}{(\rho_2-\rho_1)^2}\int_{A_{k,\rho_2}} (u_j-k)^2\,dx + C_6\text{meas}^{1-\frac{p}{\nu}}(A_{k,\rho_2}).\end{aligned}$$

We then use Lemma 4.8 to see that $\sup_{x\in B_1(y)} u_j$ has a bound independent of ε, R, θ, and y. Similarly, $\inf_{x\in B_1(y)} u_j$ has a bound independent of ε, R, θ, and y. Therefore,

$$|u_j|_\infty \leq C_7, \quad \forall\, 1\leq j\leq m, \tag{4.10}$$

where $C_7 > 0$ is a constant independent of ε, R and θ.

To estimate $|\nabla u_j|_\infty$ we rewrite equation (4.7) in the form (4.8):

$$\mathrm{div}\,[A(x,u,\nabla u)] + B(x,u,\nabla u) = 0,$$

where

$$A(x,t,\xi) = \xi - \varepsilon a(x,t,\xi)$$

and

$$B(x,t,\xi) = f(x,t) - V(x)t + \varepsilon(1+\theta)b(x,t,\xi).$$

Since $A^i(x,t,\xi) = \xi_i - \varepsilon a_i(x,t,\xi)$, it follows from Lemma 4.4(c) that, for $x, x', \xi, y \in \mathbb{R}^N$ and $t, t' \in \mathbb{R}$,

$$a^{ij}(x,t,\xi)y_i y_j = \frac{\partial A^i}{\partial \xi_j}(x,t,\xi)y_i y_j = |y|^2 - \varepsilon \frac{\partial a_i}{\partial \xi_j}(x,t,\xi)y_i y_j \geq (1 - |\varepsilon| N C_R)|y|^2,$$

$$|a^{ij}(x,t,\xi)| \leq 1 + |\varepsilon| \left| \frac{\partial a_i}{\partial \xi_j}(x,t,\xi) \right| \leq 1 + |\varepsilon| C_R,$$

and

$$|A(x,t,\xi) - A(x',t',\xi)| = |\varepsilon||a(x,t,\xi) - a(x',t',\xi)| \leq |\varepsilon| C_R(|x-x'| + |t-t'|).$$

By (V), (F_3), and Lemma 4.4(b), we have, for $x, \xi \in \mathbb{R}^N$ and $t \in [-C_7, C_7]$ where C_7 is the number from (4.10),

$$|B(x,t,\xi)| \leq C(1 + C_7^{p-1}) + \beta_0 C_7 + |\varepsilon|(1+\theta)C_R C_7. \tag{4.11}$$

Let $\varepsilon_3(R) > 0$ be such that

$$\varepsilon_3(R) < \min\left\{ \varepsilon_3''(R), \frac{1}{2NC_R} \right\}.$$

If $|\varepsilon| \leq \varepsilon_3(R)$, then all the assumptions of Lemma 4.9 are satisfied and we conclude that

$$|\nabla u_j|_\infty \leq C_8, \quad \forall\, 1 \leq j \leq m,$$

where $C_8 > 0$ is independent of ε, R and θ. Letting $C = C_7 + C_8$, we finish the proof. □

4.4 Proof of Theorem 1.3

By Lemmas 4.7 and 4.10, if $R > R_0$ and $|\varepsilon| \leq \varepsilon_3(R)$ then

$$\|u_{\varepsilon,R,\theta,j}\|_V + |u_{\varepsilon,R,\theta,j}|_{2+2\theta}^{2+2\theta} + |u_{\varepsilon,R,\theta,j}|_\infty^{1+\theta} + |\nabla u_{\varepsilon,R,\theta,j}|_\infty^2 \leq C, \quad \forall\, 1 \leq j \leq m, \tag{4.12}$$

where $C > 1$ is independent of ε, R and θ. Now we are ready to fix an R and we let

$$R_1 = \max\{R_0, C\} + 1.$$

From the arguments above, it is clear that R_0, C, and R_1 depend on $m \in \mathbb{N}$. Let $\varepsilon'_m = \varepsilon_3(R_1)$. Then, for $|\varepsilon| \le \varepsilon'_m$, $u_{\varepsilon,\theta,j} := u_{\varepsilon,R_1,\theta,j}$, $j = 1, 2, \cdots, m$, are solutions to the equation

$$-\Delta u + V(x)u = f(x,u) - \varepsilon \operatorname{div}\left[G_\xi\big(x, |u|^\theta u, \nabla u\big)\right] + \varepsilon(1+\theta)G_t\big(x, |u|^\theta u, \nabla u\big)|u|^\theta. \quad (4.13)$$

By Proposition 4.6,

$$J_{\varepsilon,\theta}(u_{\varepsilon,\theta,j+1}) - J_{\varepsilon,\theta}\big(u_{\varepsilon,\theta,j}\big) \ge 1, \quad \forall\, 1 \le j \le m-1, \quad (4.14)$$

where

$$J_{\varepsilon,\theta}(u) = \frac{1}{2}\|u\|_V^2 - \int_{\mathbb{R}^N} F(x,u)\,dx - \varepsilon \int_{\mathbb{R}^N} G\big(x, |u|^\theta u, \nabla u\big)\,dx.$$

Choose a sequence $\{\theta_k\}_{k=1}^\infty \subset (0, 2/(N-2))$ such that $\theta_k \to 0^+$. By (4.12), we may assume $u_{\varepsilon,\theta_k,j} \to u_{\varepsilon,j}$ weakly in E and a.e. in $\mathbb{R}^N$ for some $u_{\varepsilon,j} \in E$ as $k \to \infty$.

Lemma 4.11. *There exists $\varepsilon''_m \in (0, \varepsilon'_m)$ such that, for $|\varepsilon| \le \varepsilon''_m$ and up to a subsequence if necessary, we have $\nabla u_{\varepsilon,\theta_k,j} \to \nabla u_{\varepsilon,j}$ strongly in $L^2_{loc}(\mathbb{R}^N)$, and as a consequence $\nabla u_{\varepsilon,\theta_k,j} \to \nabla u_{\varepsilon,j}$ a.e. in $\mathbb{R}^N$ as $k \to \infty$.*

Proof. Write $u_k := u_{\varepsilon,\theta_k,j}$ and $u := u_{\varepsilon,j}$ for simplicity. Rewriting (4.13), we see that u_k satisfies

$$-\operatorname{div}\left[A_k(x, u_k, \nabla u_k)\right] = g_k \ \text{ in } \mathbb{R}^N,$$

where

$$A_k(x,t,\xi) := \xi - \varepsilon G_\xi\big(x, \eta_{R_1}(|t|^{1+\theta_k})|t|^{\theta_k} t, \eta_{R_1}(|\xi|^2)\xi\big)$$

and

$$g_k := f(x,u_k) - V(x)u_k + \varepsilon(1+\theta_k)G_t\big(x, |u_k|^{\theta_k} u_k, \nabla u_k\big)|u_k|^{\theta_k}.$$

By (G) there exists $C_1 > 0$ independent of k such that for $x, \xi, \xi' \in \mathbb{R}^N$ and $t \in \mathbb{R}$

$$|A_k(x,t,\xi)| \le |\xi| + |\varepsilon| C_1(|t| + |\xi|) \quad (4.15)$$

and

$$[A_k(x,t,\xi) - A_k(x,t,\xi')](\xi - \xi') \ge (1 - |\varepsilon| C_1)|\xi - \xi'|^2. \quad (4.16)$$

By (V), (F_3), and (G), choosing a larger C_1 if necessary, we have

$$|g_k| \le C_1(1 + \beta_0 + |\varepsilon|)|u_k|. \quad (4.17)$$

Fix $\varepsilon''_m \in \mathbb{R}$ such that $0 < \varepsilon''_m < \min\{\varepsilon'_m, 1/(2C_1)\}$ and let $|\varepsilon| \le \varepsilon''_m$. Let $R > 0$ and choose $\eta \in C_0^\infty(\mathbb{R}^N)$ such that

$$0 \le \eta(x) \le 1, \quad \eta(x) = 1 \text{ if } x \in B_R.$$

Then, by (4.15), (4.17) and the Rellich theorem, we have

$$\begin{aligned}
\Big| \int_{\mathbb{R}^N} \eta A_k(x,u_k,\nabla u_k)\nabla(u_k-u)\,dx\Big| \\
&\le \Big| \int_{\mathbb{R}^N} A_k(x,u_k,\nabla u_k)\nabla\eta(u_k-u)\,dx\Big| + \Big| \int_{\mathbb{R}^N} \eta g_k(u_k-u)\,dx\Big| \\
&\le C\Big(\int_{\mathbb{R}^N} (|u_k|^2+|\nabla u_k|^2)\,dx\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^N} |\nabla\eta|^2(u_k-u)^2\,dx\Big)^{\frac{1}{2}} \\
&\quad + C\Big(\int_{\mathbb{R}^N} u_k^2\,dx\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^N} \eta^2(u_k-u)^2\,dx\Big)^{\frac{1}{2}} \\
&= o(1) \qquad (4.18)
\end{aligned}$$

as $k\to\infty$. Define

$$A_0(x,t,\xi) := \xi - \varepsilon G_\xi\big(x,t,\eta_{R_1}(|\xi|^2)\xi\big).$$

Then since $\eta_{R_1}(|u_k|^{\theta_k+1}) \equiv 1$ we have

$$A_k(x,u_k,\nabla u) = \nabla u - \varepsilon G_\xi\big(x,|u_k|^{\theta_k}u_k,\eta_{R_1}(|\nabla u|^2)\nabla u\big) \to A_0(x,u,\nabla u)$$

a.e. in $\mathbb{R}^N$ as $k\to\infty$. The dominated convergence theorem together with (4.15) implies that $A_k(x,u_k,\nabla u)\to A_0(x,u,\nabla u)$ strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$. This implies

$$\begin{aligned}
\Big| \int_{\mathbb{R}^N} \eta A_k(x,u_k,\nabla u)\nabla(u_k-u)\,dx\Big| \\
&\le \Big| \int_{\mathbb{R}^N} \eta[A_k(x,u_k,\nabla u)-A_0(x,u,\nabla u)]\nabla(u_k-u)\,dx\Big| \\
&\quad + \Big| \int_{\mathbb{R}^N} \eta A_0(x,u,\nabla u)\nabla(u_k-u)\,dx\Big| \\
&= o(1) \qquad (4.19)
\end{aligned}$$

as $k\to\infty$. Using (4.16), (4.18), and (4.19) and noticing that $|\varepsilon|C_1 < \frac{1}{2}$, we obtain

$$\frac{1}{2}\int_{B_R} |\nabla(u_k-u)|^2\,dx \le \int_{\mathbb{R}^N} \eta[A_k(x,u_k,\nabla u_k)-A_k(x,u_k,\nabla u)]\nabla(u_k-u)\,dx = o(1).$$

The above estimate implies the desired result since $R>0$ is arbitrary. □

We are in a position to prove Theorem 1.3.

Proof of Theorem 1.3. Let $|\varepsilon| \le \varepsilon''_m$ and $1\le j\le m$. As in the proof of Lemma 4.11, we write $u_k := u_{\varepsilon,\theta_k,j}$ and $u := u_{\varepsilon,j}$ for simplicity. Then $u_k \to u$ weakly in E and a.e. in $\mathbb{R}^N$ and $\nabla u_k \to \nabla u$ a.e. in $\mathbb{R}^N$ as $k\to\infty$. Using these facts one can check that u is a solution of equation (1.3). Then

$$\int_{\mathbb{R}^N} \big[|\nabla u|^2 + V(x)u^2 - \varepsilon G_\xi(x,u,\nabla u)\nabla u - \varepsilon G_t(x,u,\nabla u)u\big]\,dx = \int_{\mathbb{R}^N} f(x,u)u\,dx. \qquad (4.20)$$

Since u_k is a solution of (4.13), we have

$$\int_{\mathbb{R}^N} \big[|\nabla u_k|^2 + V(x)u_k^2 - \varepsilon G_\xi(x, |u_k|^{\theta_k}u_k, \nabla u_k)\nabla u_k \\ - \varepsilon(1+\theta_k)G_t(x, |u_k|^{\theta_k}u_k, \nabla u_k)|u_k|^{\theta_k}u_k\big]\, dx = \int_{\mathbb{R}^N} f(x, u_k)u_k\, dx. \tag{4.21}$$

For simplicity of notation, denote $P(x) = f(x, u)u$, $P_k(x) = f(x, u_k)u_k$,

$$Q(x) = \frac{1}{2}|\nabla u|^2 + \frac{1}{2}V(x)u^2 - \varepsilon G_\xi(x, u, \nabla u)\nabla u - \varepsilon G_t(x, u, \nabla u)u, \tag{4.22}$$

and

$$\begin{aligned} Q_k(x) =& \frac{1}{2}|\nabla u_k|^2 + \frac{1}{2}V(x)u_k^2 \\ &- \varepsilon G_\xi(x, |u_k|^{\theta_k}u_k, \nabla u_k)\nabla u_k - \varepsilon(1+\theta_k)G_t(x, |u_k|^{\theta_k}u_k, \nabla u_k)|u_k|^{\theta_k}u_k. \end{aligned} \tag{4.23}$$

Then, by Lemma 4.11, $Q_k(x) \to Q(x)$ for a.e. $x \in \mathbb{R}^N$. By (4.12) and (G), there exists $\varepsilon_m \in (0, \varepsilon_m'')$ such that, for $|\varepsilon| \le \varepsilon_m$,

$$Q_k(x) \ge \frac{1}{4}(|\nabla u_k|^2 + V(x)u_k^2) \ge 0. \tag{4.24}$$

Using Fatou's lemma and (4.20), (4.21) and (4.24), we estimate as

$$\begin{aligned} \int_{\mathbb{R}^3} P(x)\, dx =& \frac{1}{2}\|u\|_V^2 + \int_{\mathbb{R}^3} Q(x)\, dx \le \frac{1}{2}\varliminf_{k\to\infty}\|u_k\|_V^2 + \varliminf_{k\to\infty}\int_{\mathbb{R}^3} Q_k(x)\, dx \\ \le& \frac{1}{2}\varlimsup_{k\to\infty}\|u_k\|_V^2 + \varliminf_{k\to\infty}\int_{\mathbb{R}^3} Q_k(x)\, dx \le \lim_{k\to\infty}\Big(\frac{1}{2}\|u_k\|_V^2 + \int_{\mathbb{R}^3} Q_k(x)\, dx\Big) \\ =& \lim_{k\to\infty}\int_{\mathbb{R}^3} P_k(x)\, dx = \int_{\mathbb{R}^3} P(x)\, dx. \end{aligned}$$

We see that $\|u_k\|_V \to \|u\|_V$ and therefore $u_k \to u$ strongly in E.

Recalling $u_k := u_{\varepsilon,\theta_k,j}$ and $u := u_{\varepsilon,j}$ and using the strong convergence, we have

$$\lim_{k\to\infty} J_{\varepsilon,\theta_k}\big(u_{\varepsilon,\theta_k,j}\big) = J_\varepsilon\big(u_{\varepsilon,j}\big).$$

This together with (4.14) implies that $\{u_{\varepsilon,j}\}_{j=1}^m$ are m distinct solutions of (1.3). We complete the proof. □

5 Proof of Theorem 1.5

We prove Theorem 1.5 in this section. First we have the following lemma.

Lemma 5.1. *There exists a constant $C > 0$ such that for any $u \in H^1(\mathbb{R}^3)$*

(a) $\|\phi_u\|_{D^{1,2}(\mathbb{R}^3)} \le C|u|_{12/5}^2$, $\int_{\mathbb{R}^3} \phi_u u^2\, dx \le C|u|_{12/5}^4$,

(b) $\int_{\mathbb{R}^3} |u|^3\,dx \le \int_{\mathbb{R}^3} |\nabla u|^2\,dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2\,dx$,

(c) *if* $u_n \to u$ *in* $L^{12/5}(\mathbb{R}^3)$, *then* $\phi_{u_n} \to \phi_u$ *in* $D^{1,2}(\mathbb{R}^3)$, $\int_{\mathbb{R}^3} \phi_{u_n} u_n^2\,dx \to \int_{\mathbb{R}^3} \phi_u u^2\,dx$, *and* $\int_{\mathbb{R}^3} \phi_{u_n} u_n v\,dx \to \int_{\mathbb{R}^3} \phi_u uv\,dx$ *for* $v \in H^1(\mathbb{R}^3)$.

Proof. We have

$$\int_{\mathbb{R}^3} |u|^3\,dx = \int_{\mathbb{R}^3} \nabla\phi_u \nabla(|u|)\,dx \le \int_{\mathbb{R}^3} |\nabla u|^2\,dx + \frac{1}{4}\int_{\mathbb{R}^3} |\nabla\phi_u|^2\,dx.$$

This proves (b) since

$$\int_{\mathbb{R}^3} \phi_u u^2\,dx = \int_{\mathbb{R}^3} |\nabla\phi_u|^2\,dx.$$

The proofs of (a) and (c) are easy. □

According to the principle of symmetric criticality [25], in the remaining of this section, we shall work in $H_r^1(\mathbb{R}^N)$. The formal functional associated with (1.4) is

$$J_\varepsilon(u) = \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2\,dx + \frac{1}{\varepsilon^2}\int_{\mathbb{R}^3} G(x,\varepsilon u)\,dx.$$

Here, we choose the norm

$$\|u\| := \Big(\int_{\mathbb{R}^3} (|\nabla u|^2 + \omega u^2)\,dx\Big)^{\frac{1}{2}}$$

in $H_r^1(\mathbb{R}^3)$ and $G(x,t) = \int_0^t g(x,s)\,ds$. We set

$$J(u) := \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2\,dx \quad \text{for } u \in H_r^1(\mathbb{R}^3).$$

We first consider J. Clearly, $J \in C^1(H_r^1(\mathbb{R}^3),\mathbb{R})$. As a direct consequence of Lemma 5.1, we have the next lemma.

Lemma 5.2. *There exists a constant* $C > 0$ *such that for any* $u \in H_r^1(\mathbb{R}^3)$

(a) $J(u) \ge \frac{1}{2}\|u\|^2 - C\|u\|^4$, *and*

(b) $J(u) \le \frac{1}{2}\|u\|^2 + \int_{\mathbb{R}^3} |\nabla u|^2\,dx - \int_{\mathbb{R}^3} |u|^3\,dx$.

Lemma 5.3. $J : H_r^1(\mathbb{R}^3) \to \mathbb{R}$ *satisfies the* $(PS)_c$ *condition for every* $c \in \mathbb{R}$.

Proof. Let $\{u_n\} \subset H_r^1(\mathbb{R}^3)$ be a $(PS)_c$ sequence. Then for n large

$$c + 1 + \|u_n\| \ge J(u_n) - \frac{1}{4}\langle J'(u_n), u_n\rangle = \frac{1}{4}\|u_n\|^2.$$

Hence $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^3)$. Assume that $u_n \to u$ weakly in $H_r^1(\mathbb{R}^3)$, strongly in $L^{12/5}(\mathbb{R}^3)$, and a.e. in $\mathbb{R}^3$. Using Lemma 5.1 one can easily verify that $u_n \to u$ strongly in $H_r^1(\mathbb{R}^3)$. The proof is finished. □

By Lemmas 5.2 and 5.3, we see that all the conditions of Theorem 4.1 are satisfied. By Theorem 4.1 we obtain the next result.

Lemma 5.4. *The functional $J : H_r^1(\mathbb{R}^3) \to \mathbb{R}$ has a sequence of essential values c_j with $c_j \to +\infty$ as $j \to \infty$.*

Now we fix a number $p \in (2, 6)$. Let $\theta \in (0, 1)$ be such that

$$p(1+\theta) < 6.$$

For $R > 1$, let η_R be the cut off function defined in Section 3.2. For $0 < \varepsilon < 1$, we consider the modified functional

$$J_{\varepsilon,R,\theta}(u) = \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2\, dx + \frac{1}{\varepsilon^2}\eta_R(|u|_{2+2\theta}^{2+2\theta} + |u|_{p(1+\theta)}^{p(1+\theta)}) \int_{\mathbb{R}^3} G(x, \varepsilon\eta_R(|u|^{1+\theta})|u|^\theta u)\, dx.$$

It is easy to check that $J_{\varepsilon,R,\theta} \in C^1(H_r^1(\mathbb{R}^3), \mathbb{R})$, and for $u, v \in H_r^1(\mathbb{R}^3)$

$$\begin{aligned}
\langle J'_{\varepsilon,R,\theta}(u), v\rangle = &\int_{\mathbb{R}^3} (\nabla u \nabla v + \omega uv)\, dx - \int_{\mathbb{R}^3} \phi_u uv\, dx \\
&+ \frac{1}{\varepsilon^2}\eta'_R(|u|_{2+2\theta}^{2+2\theta} + |u|_{p(1+\theta)}^{p(1+\theta)}) \int_{\mathbb{R}^3} G(x, \varepsilon\eta_R(|u|^{1+\theta})|u|^\theta u)\, dx \\
&\times \Big[(2+2\theta)\int_{\mathbb{R}^3} |u|^{2\theta} uv\, dx + p(1+\theta)\int_{\mathbb{R}^3} |u|^{p(1+\theta)-2} uv\, dx\Big] \\
&+ \frac{1}{\varepsilon}(1+\theta)\eta_R(|u|_{2+2\theta}^{2+2\theta} + |u|_{p(1+\theta)}^{p(1+\theta)}) \\
&\times \int_{\mathbb{R}^3} g(x, \varepsilon\eta_R(|u|^{1+\theta})|u|^\theta u)(\eta'_R(|u|^{1+\theta})|u|^{1+2\theta} v + \eta_R(|u|^{1+\theta})|u|^\theta v)\, dx.
\end{aligned}$$

Lemma 5.5. *For $R > 1$ there exists $C_R > 0$ independent of ε and θ such that*

$$\sup_{u \in H_r^1(\mathbb{R}^3)} |J_{\varepsilon,R,\theta}(u) - J(u)| \le \frac{1}{R} + \varepsilon^{p-2} C_R.$$

Proof. By (g_2), for $R > 1$, there exists $C_R > 0$ such that $|G(x,t)| \le \frac{1}{R^2}t^2 + C_R|t|^p$ for $x \in \mathbb{R}^3$ and $|t| \le R$. Then for any $u \in H_r^1(\mathbb{R}^3)$,

$$\begin{aligned}
|J_{\varepsilon,R,\theta}(u) - J(u)| =& \frac{1}{\varepsilon^2}\eta_R\big(|u|_{2+2\theta}^{2+2\theta} + |u|_{p(1+\theta)}^{p(1+\theta)}\big) \Big| \int_{\mathbb{R}^3} G(x, \varepsilon\eta_R(|u|^{1+\theta})|u|^\theta u)\, dx \Big| \\
\le& \eta_R\big(|u|_{2+2\theta}^{2+2\theta} + |u|_{p(1+\theta)}^{p(1+\theta)}\big)\Big(\frac{1}{R^2}|u|_{2+2\theta}^{2+2\theta} + C_R\varepsilon^{p-2}|u|_{p(1+\theta)}^{p(1+\theta)}\Big) \\
\le& \frac{1}{R} + \varepsilon^{p-2} C_R.
\end{aligned}$$

The result follows. ∎

Lemma 5.6. *Let ε, R and θ be fixed. Then $J_{\varepsilon,R,\theta} : H_r^1(\mathbb{R}^3) \to \mathbb{R}$ satisfies the $(PS)_c$ condition for every $c \in \mathbb{R}$.*

Proof. We compute as in the proof of Lemma 5.5 to show that

$$J_{\varepsilon,R,\theta}(u) - \frac{1}{4}\langle J'_{\varepsilon,R,\theta}(u), u\rangle \geq \frac{1}{4}\|u\|^2 - \frac{C}{R} - \varepsilon^{p-2}C_R, \quad \forall u \in H_r^1(\mathbb{R}^3). \tag{5.1}$$

Let $\{u_n\} \subset H_r^1(\mathbb{R}^3)$ be a $(PS)_c$ sequence of $J_{\varepsilon,R,\theta}$. Then $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^3)$. Assume that $u_n \to u$ weakly in $H_r^1(\mathbb{R}^3)$ and a.e. in $\mathbb{R}^3$. Then using (g_2), Lemma 5.1 and the fact that $2 < 2(1+\theta) < p(1+\theta) < 6$ one can easily verify that u is a critical point of $J_{\varepsilon,R,\theta}$ and $u_n \to u$ strongly in $H_r^1(\mathbb{R}^3)$. □

By Lemmas 2.4–2.5 and Lemmas 5.4–5.6, we obtain the next result.

Lemma 5.7. *There exist $R_0 > 1$ and, for $R \geq R_0$, $\varepsilon_0(R) > 0$ independent of θ and j such that if $R \geq R_0$ and $0 < \varepsilon \leq \varepsilon_0(R)$ then $J_{\varepsilon,R,\theta}$ has m distinct critical values $\{c_{\varepsilon,R,\theta,j}\}_{j=1}^m$ satisfying*

$$0 < c_0^{-1} \leq c_{\varepsilon,R,\theta,j} \leq c_0, \quad 1 \leq j \leq m$$

and

$$c_{\varepsilon,R,\theta,j+1} - c_{\varepsilon,R,\theta,j} \geq 1, \quad 1 \leq j \leq m-1,$$

where $c_0 = c_0(m) > 0$ is independent of ε, R, θ and j.

We remark that, by (5.1), we may decrease $\varepsilon_0(R)$ in Lemma 5.7 if necessary so that

$$J_{\varepsilon,R,\theta}(u) - \frac{1}{4}\langle J'_{\varepsilon,R,\theta}(u), u\rangle \geq \frac{1}{4}\|u\|^2 - C, \quad \forall u \in H_r^1(\mathbb{R}^3), \tag{5.2}$$

for some constant $C > 0$ independent of ε, R, θ. Let $\{u_{\varepsilon,R,\theta,j}\}_{j=1}^m \subset H_r^1(\mathbb{R}^3)$ be m distinct critical points of $J_{\varepsilon,R,\theta}$ obtained by Lemma 5.7 with $J_{\varepsilon,R,\theta}(u_{\varepsilon,R,\theta,j}) = c_{\varepsilon,R,\theta,j}$. Then, by Lemma 5.7, (5.2), and the Sobolev inequalities, we have

Lemma 5.8. *There exists $C_1 > 0$ independent of ε, R, θ, and j such that*

$$\|u_{\varepsilon,R,\theta,j}\| + |u_{\varepsilon,R,\theta,j}|_{2+2\theta}^{2+2\theta} + |u_{\varepsilon,R,\theta,j}|_{p(1+\theta)}^{p(1+\theta)} \leq C_1, \quad 1 \leq j \leq m.$$

Lemma 5.9. *There exist $R_1 > R_0$ and, for $R \geq R_1$, $\varepsilon_1(R) \in (0, \varepsilon_0(R))$ independent of θ and j such that if $R \geq R_1$ and $0 < \varepsilon \leq \varepsilon_1(R)$ then*

$$|u_{\varepsilon,R,\theta,j}|_\infty \leq C_2, \quad 1 \leq j \leq m,$$

where $C_2 > 0$ is a constant independent of ε, R, θ, and j.

Proof. For simplicity we set $u := u_{\varepsilon,R,\theta,j}$ and $\phi := \phi_{u_{\varepsilon,R,\theta,j}}$. Let C_1 be the constant from Lemma 5.8 and

$$R_1 = \max\{R_0, C_1\} + 1.$$

Let $R \geq R_1$. Then u satisfies, for any $v \in H^1(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} (\nabla u \nabla v + \omega uv)\, dx - \int_{\mathbb{R}^3} \phi uv\, dx$$
$$+\frac{1}{\varepsilon}(1+\theta) \int_{\mathbb{R}^3} g(x, \varepsilon\eta_R(|u|^{1+\theta})|u|^\theta u)(\eta_R'(|u|^{1+\theta})|u|^{1+2\theta} + \eta_R(|u|^{1+\theta})|u|^\theta) v\, dx = 0.$$

We first estimate $|\phi|_\infty$. Choosing $|\phi^M|^\alpha \phi^M$ for $M > 0$ and $\alpha > 1$ as a test function in the Poisson equation

$$-\Delta\phi = u^2,$$

we deduce that

$$\begin{aligned}(\alpha+1)\int_{\mathbb{R}^3} |\phi^M|^\alpha |\nabla(\phi^M)|^2\, dx &= \int_{\mathbb{R}^3} u^2 |\phi^M|^\alpha \phi^M\, dx \\ &\leq \Big(\int_{\mathbb{R}^3} u^4\, dx\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^3} |\phi^M|^{2\alpha+2}\, dx\Big)^{\frac{1}{2}} \\ &\leq C\Big(\int_{\mathbb{R}^3} |\phi^M|^{2\alpha+2}\, dx\Big)^{\frac{1}{2}},\end{aligned}$$

where $C > 0$ is independent of $\varepsilon,\ R,\ \theta$ and j by Lemma 5.8. It follows that

$$\Big(\int_{\mathbb{R}^3} |\phi^M|^{3\alpha+6}\, dx\Big)^{\frac{1}{3}} \leq C(\alpha+1)^2 \int_{\mathbb{R}^3} |\phi^M|^\alpha |\nabla(\phi^M)|^2\, dx \leq C\alpha \Big(\int_{\mathbb{R}^3} |\phi^M|^{2\alpha+2}\, dx\Big)^{\frac{1}{2}}.$$

Letting $M \to \infty$ we have

$$|\phi|_{3\alpha+6} \leq (C\alpha)^{\frac{1}{\alpha+2}} |\phi|_{2\alpha+2}^{\frac{\alpha+1}{\alpha+2}}, \tag{5.3}$$

where $C > 0$ is independent of $\varepsilon,\ R,\ \theta$ and j. If we set

$$\alpha_k = 6\Big(\frac{3}{2}\Big)^k - 4, \quad k = 0, 1, 2, \cdots,$$

then by (5.3) we obtain

$$|\phi|_{2\alpha_k+2} \leq (C\alpha_{k-1})^{\frac{1}{\alpha_{k-1}+2}} |\phi|_{2\alpha_{k-1}+2}^{\frac{\alpha_{k-1}+1}{\alpha_{k-1}+2}}, \quad k = 1, 2, \cdots.$$

An iteration process implies that

$$|\phi|_{2\alpha_k+2} \leq \Big[\prod_{j=0}^{k-1} (C\alpha_j)^{\frac{1}{\alpha_j+2}\left(\prod_{j+1\leq i\leq k-1} \frac{\alpha_i+1}{\alpha_i+2}\right)}\Big] |\phi|_6^{\prod_{j=0}^{k-1} \frac{\alpha_j+1}{\alpha_j+2}}.$$

This implies, for any $k \in \mathbb{N}$,

$$|\phi|_{2\alpha_k+2} \leq C(1 + |\phi|_6) \leq C(1 + \|u\|^2).$$

Letting $k \to \infty$ and using Lemma 5.8 we have, for some constant $C > 0$ independent of $\varepsilon,\ R,\ \theta$ and j,

$$|\phi|_\infty \leq C. \tag{5.4}$$

Next we estimate $|u|_\infty$. Recall that u satisfies

$$\int_{\mathbb{R}^3} (\nabla u \nabla v + \omega uv)\, dx = \int_{\mathbb{R}^3} \phi uv\, dx + \int_{\mathbb{R}^3} g_{\varepsilon,R,\theta}(x,u)v\, dx \quad \text{for } v \in H^1(\mathbb{R}^3), \tag{5.5}$$

where

$$g_{\varepsilon,R,\theta}(x,t) = -\frac{1}{\varepsilon}(1+\theta)g(x, \varepsilon\eta_R(|t|^{1+\theta})|t|^\theta t)(\eta_R'(|t|^{1+\theta})|t|^{1+2\theta} + \eta_R(|t|^{1+\theta})|t|^\theta).$$

It can be easily seen that these exists $0 < \varepsilon_1(R) < \varepsilon_0(R)$ such that for $R \geq R_1$ and $0 < \varepsilon \leq \varepsilon_1(R)$,

$$|g_{\varepsilon,R,\theta}(x,t)| \leq C|t| \quad \text{for all } (x,t) \in \mathbb{R}^3 \times \mathbb{R},$$

where $C > 0$ is a constant independent of ε, R, θ. This together with (5.4)–(5.5) implies that there exists $C > 0$ independent of ε, R, θ such that for $R \geq R_1$ and $0 < \varepsilon \leq \varepsilon_1(R)$,

$$\Big| \int_{\mathbb{R}^3} (\nabla u \nabla v + \omega uv)\, dx \Big| \leq C \int_{\mathbb{R}^3} |u||v| \quad \text{for all } v \in H^1(\mathbb{R}^3). \tag{5.6}$$

Choosing $v = |u^M|^\alpha u^M$ for $M > 0$ and $\alpha > 1$ as a test function in (5.6), we obtain

$$|u|_{3\alpha+6} \leq (C\alpha)^{\frac{1}{\alpha+2}} |u|_{\alpha+2},$$

where $C > 0$ is independent of $\varepsilon,\ R,\ \theta$ and j. Then a Moser iteration process as above together with Lemma 5.8 implies that

$$|u|_\infty \leq C|u|_6 \leq C\|u\| \leq C,$$

where $C > 0$ is independent of $\varepsilon,\ R,\ \theta$ and j. This completes the proof. □

Proof of Theorem 1.5. We can fix an R_2 such that

$$R_2 > \max\{R_1,\ (C_2+1)^2\},$$

where C_2 is the constant from Lemma 5.9. Let $\varepsilon_m = \varepsilon_1(R_2)$. For $|\varepsilon| \in (0, \varepsilon_m)$, denote $u_{\varepsilon,\theta,j} := u_{\varepsilon,R_2,\theta,j}$ and define for $u \in H^1(\mathbb{R}^3)$,

$$J_{\varepsilon,\theta}(u) := \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2\, dx + \frac{1}{\varepsilon^2}\int_{\mathbb{R}^3} G(x,\ \varepsilon|u|^\theta u)\, dx.$$

Then by Lemmas 5.7–5.9, $u_{\varepsilon,\theta,j}$ satisfies

$$0 < c_0^{-1} \leq J_{\varepsilon,\theta}(u_{\varepsilon,\theta,j}) \leq c_0, \quad 1 \leq j \leq m, \tag{5.7}$$

$$J_{\varepsilon,\theta}(u_{\varepsilon,\theta,j}) + 1 \leq J_{\varepsilon,\theta}(u_{\varepsilon,\theta,j+1}), \quad 1 \leq j \leq m-1, \tag{5.8}$$

$$\begin{aligned}\int_{\mathbb{R}^3} (\nabla u_{\varepsilon,\theta,j}\nabla v + \omega u_{\varepsilon,\theta,j} v)\, dx - \int_{\mathbb{R}^3} \phi_{u_{\varepsilon,\theta,j}} u_{\varepsilon,\theta,j} v\, dx \\ + \frac{1}{\varepsilon}(1+\theta)\int_{\mathbb{R}^3} g(x,\ \varepsilon|u_{\varepsilon,\theta,j}|^\theta u_{\varepsilon,\theta,j})|u_{\varepsilon,\theta,j}|^\theta v = 0, \quad \forall\, v \in H^1(\mathbb{R}^3),\end{aligned} \tag{5.9}$$

$$\|u_{\varepsilon,\theta,j}\| + |u_{\varepsilon,\theta,j}|_\infty \le C, \quad 1 \le j \le m, \tag{5.10}$$

where $C > 0$ is a constant independent of ε, θ, and j. We may assume that $u_{\varepsilon,\theta,j} \to u_{\varepsilon,j}$ weakly in $H_r^1(\mathbb{R}^3)$ and a.e. in $\mathbb{R}^3$ as $\theta \to 0^+$. Using (5.7)–(5.10) we can verify that $u_{\varepsilon,\theta,j} \to u_{\varepsilon,j}$ strongly in $H_r^1(\mathbb{R}^3)$, $u_{\varepsilon,j}$ $(j = 1, 2, \cdots, m)$ are solutions of (1.4), and

$$\lim_{\theta \to 0^+} J_{\varepsilon,\theta}(u_{\varepsilon,\theta,j}) = J_\varepsilon(u_{\varepsilon,j}), \quad 1 \le j \le m.$$

Then

$$J_\varepsilon(u_{\varepsilon,j}) + 1 \le J_\varepsilon(u_{\varepsilon,j+1}), \quad 1 \le j \le m-1,$$

and $(u_{\varepsilon,j})_{j=1}^m$ are m distinct nontrivial solutions of (1.4). □

6 Concluding remarks and additional results

In this section, we first give some variants of Theorem 1.3. In Theorem 1.3, we have assumed

$$\beta_0 := \sup_{x \in \mathbb{R}^N} V(x) < +\infty. \tag{6.1}$$

This assumption is only used to deduce (4.11) in the proof of Lemma 4.10 in order to apply Lemma 4.9 from [20], and to deduce (4.17) in the proof of Lemma 4.11 in order to have estimate as in (4.18). According to the discussions on [20, Page 347], Lemma 4.9 still holds if the boundedness assumption on $B(x, t, \xi)$ is relaxed by assuming

$$\sup_{x,\, \xi \in \mathbb{R}^N,\, |t| \le M_0} \int_{B_1(x)} |B(y, t, \xi)|^q dy < +\infty \quad \text{for some } q > N.$$

Accordingly, the proof of Lemma 4.10 is valid if condition (6.1) is replaced with the weaker one:

$$\sup_{x \in \mathbb{R}^N} \int_{B_1(x)} |V(y)|^q dy < +\infty \quad \text{for some } q > N. \tag{6.2}$$

In addition, under assumption (6.2), instead of (4.17) we have

$$|g_k| \le (V(x) + C)|u_k|$$

and the second term in (4.18) can be estimated as

$$\begin{aligned}&\Big|\int_{\mathbb{R}^N} \eta g_k (u_k - u)\, dx\Big| \\ &\quad \le \Big(\int_{\mathbb{R}^N} \eta (V + C)^q\, dx\Big)^{\frac{1}{q}} \Big(\int_{\mathbb{R}^N} \eta u_k^{\frac{2q}{q-2}}\, dx\Big)^{\frac{q-2}{2q}} \Big(\int_{\mathbb{R}^N} \eta (u_k - u)^2\, dx\Big)^{\frac{1}{2}} = o(1).\end{aligned}$$

This means the proof of Lemma 4.11 is also valid. Therefore, we have the following result.

Theorem 6.1. *Theorem 1.3 is valid when condition (6.1) is replaced with (6.2).*

In the setting of Theorem 1.3, if $N = 4$ or $N \geq 6$ then a method initiated by Bartsch and Willem in [5] can be used to prove existence of multiple nonradial solutions. Let us sketch the idea from [5] (see also [31]). Let $2 \leq m \leq N/2$ be a fixed integer different from $(N-1)/2$. The action of

$$G := \mathbf{O}(m) \times \mathbf{O}(m) \times \mathbf{O}(N-2m)$$

on $H^1(\mathbb{R}^N)$ is defined by

$$gu(x) := u(g^{-1}x).$$

Let τ be the involution defined on $\mathbb{R}^N = \mathbb{R}^m \oplus \mathbb{R}^m \oplus \mathbb{R}^{N-2m}$ by

$$\tau(x_1, x_2, x_3) := (x_2, x_1, x_3).$$

The action of $H := \{\mathrm{id}, \tau\}$ on $H^1(\mathbb{R}^N)$ is defined by

$$hu(x) := \begin{cases} u(x), & \text{if } h = \mathrm{id}, \\ -u(h^{-1}x), & \text{if } h = \tau. \end{cases}$$

Define

$$X := \{u \in H^1(\mathbb{R}^N) \,|\, gu = u, \forall g \in G, \ \forall g \in H\}. \tag{6.3}$$

With the norm $\|\cdot\|_V$ defined at the beginning of Section 4, X is compactly embedded into $L^p(\mathbb{R}^N)$ for any $2 < p < 2^*$. Working in X as in [5], along the lines of discussions in Section 4, we have the following theorem.

Theorem 6.2. *Let $N = 4$ or $N \geq 6$. Assume (V), (F_1)–(F_5) and (G). Assume also that $G(x, t, \xi)$ is even in t. Then for any $m \in \mathbb{N}$ there exists $\varepsilon_m > 0$ such that (1.3) has at least m distinct nonradial solutions provided $|\varepsilon| \leq \varepsilon_m$.*

We assumed that $V(x) = V(|x|)$, $f(x,t) = f(|x|,t)$, $f(x,-t) = -f(x,t)$, and $G(x,t,\xi) = G(|x|,t,|\xi|)$ in Theorems 1.3 and 6.1 and in addition that $G(x,-t,\xi) = G(x,t,\xi)$ in Theorem 6.2, and obtained multiple radial solutions and multiple nonradial solutions, respectively. These symmetry assumptions lead to compactness for Palais-Smale sequences and make the functional J_ε together with its modified versions $J_{\varepsilon,R,\theta}$ and $J_{\varepsilon,\theta}$ defined in Section 4 to be invariant with respect to the $\mathbf{O}(N)$ action in the case of Theorems 1.3 and 6.1 and the G and H actions in the case of Theorem 6.2. Invariance of the functionals makes it possible for us to work in $H^1_r(\mathbb{R}^N)$ or X defined in (6.3) which are compactly embedded into $L^p(\mathbb{R}^N)$ for any $2 < p < 2^*$ and then to apply the principle of symmetric criticality [25].

We now consider a case in which G is independent of ξ and in which the equation has no spherical symmetry with respect to $x \in \mathbb{R}^N$. Instead of condition (6.1) or (6.2) on V, we assume that, for any $M > 0$ and $r > 0$,

$$\mathrm{meas}(\{x \in B_r(y) : V(x) \leq M\}) \to 0 \quad \text{as } |y| \to \infty. \tag{6.4}$$

Clearly, (6.4) holds if V is coercive, that is, if $V(x) \to +\infty$ as $|x| \to \infty$. By a result of Molčanov [22] (see also [17, Corollary 6.2]), (6.4) implies that the space

$$X := \left\{ u \in H^1(\mathbb{R}^N) \,\middle|\, \int_{\mathbb{R}^N} V(x)u^2\,dx < \infty \right\} \tag{6.5}$$

with the norm

$$\|u\|_V = \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\,dx \right)^{\frac{1}{2}}$$

is compactly embedded into $L^p(\mathbb{R}^N)$ for any $2 \le p < 2^*$. This compactness property of the embedding maps holds for any $N \ge 1$. Working in the space X as defined in (6.5), we can prove the following theorem, in which we do not impose any constraint on dimension N. That is, the theorem holds for any $N \ge 1$.

Theorem 6.3. *Assume* (F_2)–(F_5) *and*

(V') $V \in C(\mathbb{R}^N, \mathbb{R})$, $\alpha_0 := \inf_{x \in \mathbb{R}^N} V(x) > 0$, V *satisfies* (6.4)*;*

(F_1') $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, $f(x,t)$ *is odd in* t*;*

(G') $G \in C^1(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, $G(x,0) \equiv 0$, *and for any* $R > 0$ *there exists* $C_R > 0$ *such that, for* $x \in \mathbb{R}^N$ *and* $|t| \le R$,

$$|G_t(x,t)| \le C_R|t|.$$

Then for any $m \in \mathbb{N}$ *there exists* $\varepsilon_m > 0$ *such that* (1.3) *has at least* m *distinct solutions provided* $|\varepsilon| \le \varepsilon_m$.

Proof. We argue along the lines of the approach in Section 4, working now in the space X defined in (6.5) instead of $H_r^1(\mathbb{R}^N)$. For this proof, the discussions in Section 4 are similar but simpler. Since G is independent of ξ, we have $a(x,t,\xi) \equiv 0$ and we need neither an estimate of $|\nabla u_{\varepsilon,R,\theta,j}|_\infty$ as in Lemma 4.10 nor a result as in Lemma 4.11. In (4.22) and (4.23), we need to replace the definitions of $Q(x)$ and $Q_k(x)$ with

$$Q(x) = \frac{1}{2}V(x)u^2 - \varepsilon G_t(x,u)u$$

and

$$Q_k(x) = \frac{1}{2}V(x)u_k^2 - \varepsilon(1+\theta_k)G_t(x,|u_k|^{\theta_k}u_k)|u_k|^{\theta_k}u_k$$

respectively. The rest is almost the same as in Section 4. □

At last, we give variants of Theorem 1.5 and Corollary 1.6. Consider the Choquard equation with a potential $V(x)$

$$\begin{cases} -\Delta u + V(x)u - \phi_u u + \frac{1}{\varepsilon}g(x,\varepsilon u) = 0 & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3). \end{cases} \tag{6.6}$$

There is no spherical symmetry with respect to $x \in \mathbb{R}^N$ needed for this equation. As in Section 1, letting $\lambda = \frac{1}{\varepsilon^2}$, (6.6) is equivalent to the Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u - \lambda\phi u + g(x,u) = 0 \quad \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 \quad \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \quad \phi \in D^{1,2}(\mathbb{R}^3). \end{cases} \tag{6.7}$$

The same argument as in Section 5 can be used to prove the following theorem.

Theorem 6.4. *Assume*

(V'') $V \in C(\mathbb{R}^3, \mathbb{R})$, $\alpha_0 := \inf_{x\in\mathbb{R}^N} V(x) > 0$, V *satisfies* (6.4);

(g') $g \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$, $\lim_{t\to 0} g(x,t)/t = 0$ *uniformly for* $x \in \mathbb{R}^3$, *and for any* $R > 0$ *there exists* $C_R > 0$ *such that if* $x \in \mathbb{R}^3$ *and* $|t| \le R$ *then* $|g(x,t)| \le C_R$.

Then for any given $m \in \mathbb{N}$ *there exists* $\varepsilon_m > 0$ *such that* (6.6) *has at least* m *distinct solutions provided* $|\varepsilon| \le \varepsilon_m$.

Corollary 6.5. *Suppose that* (V'') *and* (g') *hold. Then for any given* $m \in \mathbb{N}$ *there exists* $\Lambda_m > 0$ *such that, for* $\lambda \ge \Lambda_m$, (6.7) *has at least* m *distinct solutions* (u, ϕ).

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